ON THE STRUCTURE OF THE RECURSIVELY ENUMERABLE DEGREES

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Fakultät für Mathematik
an der Ludwigs-Maximilians-Universität München

vorgelegt von

KLAUS AMBOS-SPIES

aus Mannheim

im Oktober 1980

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INTRODUCTION

Recursion theory had its origin in the 1930s when several formalizations of the intuitive notions of computability and decidability were developed.

A partial function from the set ω of nonnegative integers to ω is computable if an effective procedure, i.e., an algorithm, exists which for any given x in the domain of f yields f(x) in finitely many steps, and which gives no output if $x \notin \mathrm{dom}(f)$. Analogously, a subset A of ω is decidable if there exists an effective procedure to decide for any number if it is an element of A, i.e., the set A is decidable iff its characteristic function is computable.

Though the methods of formalizing computability were quite different (examples are Turing-machines, Markov-algorithms, equation calculi and recursion schemes combined with the μ-operator), all formalizations yielded the same class of functions, which is called the class of recursive functions. Recursive sets are those which have a recursive characteristic function.

Church's Thesis, which is commonly accepted, asserts that the formal notion recursive and the intuitive notions computable and decidable coincide.

Besides recursiveness the central notion in recursion theory is recursive enumerability. A set is recursively enumerable (r.e.) if it is the range of a partial recursive function. Intuitively, A is r.e. if there is an effective procedure which enumerates A.

While obviously every recursive set is r.e., the converse is not true. This is illustrated by the undecidability of first order logic and Peano arithmetic: The sets of the provable formulas (coded by numbers) of these systems are r.e. but not recursive.

A set A is said to be T-reducible to B (or to be recursive in B), in symbols $A \leq_T B$, if it is possible to decide for any number x whether x belongs to A, using a procedure which is effective with the exception that information about the membership relation of B is supposed to be known and may be used. Again different formalizations of this reduction turned out to be equivalent, and by the Relativized Church's Thesis they coincide with the described intuitive notion of reduction. The relation $\leq_T$ is reflexive and transitive. We write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$ and say that A and B are T-equivalent. If A and B are T-equivalent then they have the same degree of undecidability, i.e., a (hypothetical) decision procedure for A yields one for B and conversely. Hence the equivalence classes of $\equiv_T$ are called degrees (of unsolvability). The degree which contains A is denoted by dg(A), the partial ordering of the degrees which is induced by $\leq_T$ is denoted by $\leq$. Since a degree contains all sets which are "equally" undecidable, and lesser degrees contain "less" undecidable sets, there exists a least degree $\underset{\sim}{0}$ which contains just the recursive sets.

Of particular interest are the degrees which contain r.e. sets. These are called recursively enumerable (r.e.) degrees. The study of the structure $(\underset{\sim}{R}, \leq)$ of the r.e. degrees is one of the central fields of recursion theory.

By the enumeration theorem for r.e. sets, there exists a largest degree $\underset{\sim}{0}'$ in $\underset{\sim}{R}$. Another elementary fact is that any two r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have a supremum $\underset{\sim}{a} \cup \underset{\sim}{b}$ (In the following $\underset{\sim}{a}, \underset{\sim}{b}$, etc. will always denote r.e. degrees.).

In 1944 Post raised the question - known as Post's Problem - if there exist other r.e. degrees than $\underset{\sim}{0}$ and $\underset{\sim}{0}'$. In 1956 Friedberg and Muchnik answered the question positively by proving the existence of incomparable r.e. degrees. The technique they used is called the priority

method. It proved to be fundamental for the investigation of the the structure of $\underset{\sim}{R}$. Numerous improvements and variations of it have been developed since then.

Slight variations of the original Friedberg-Muchnik technique sufficed to prove a series of results, e.g. that there are infinitely many pairwise incomparable r.e. degrees. In the first half of the '60s, using much more powerful variants of the priority method, Sacks showed that every nonzero r.e. degree can be split into two lesser ones (i.e. $\forall \underset{\sim}{a} > \underset{\sim}{0}\ \exists \underset{\sim}{b}_0, \underset{\sim}{b}_1$ ($\underset{\sim}{a} = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ and $\underset{\sim}{b}_0, \underset{\sim}{b}_1 < \underset{\sim}{a}$)) and that the r.e. degrees are dense (i.e. $\forall \underset{\sim}{a}, \underset{\sim}{b}$ ($\underset{\sim}{a} < \underset{\sim}{b} \Rightarrow \exists \underset{\sim}{c}$ ($\underset{\sim}{a} < \underset{\sim}{c} < \underset{\sim}{b}$))).

These results are the culmination of a first period in investigating $\underset{\sim}{R}$. All results derived in this period indicated that the structure of $\underset{\sim}{R}$ is homogeneous (i.e. that all r.e. degrees $\neq \underset{\sim}{0}, \underset{\sim}{0}'$ have the same properties). This inspired Shoenfield's Conjecture (1965) which asserts that the r.e. degrees are a certain homogeneous structure. The most interesting consequence of the conjecture was that no pair of incomparable r.e. degrees has an infimum. Lachlan and Yates investigated this question and showed that

(1) there exist incomparable r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that the infimum $\underset{\sim}{a} \cap \underset{\sim}{b}$ of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ doesn't exist,

but also that

(2) there exists a minimal pair of r.e. degrees, i.e. a pair $(\underset{\sim}{a}, \underset{\sim}{b})$ of incomparable r.e. degrees s.t. $\underset{\sim}{a} \cap \underset{\sim}{b} = \underset{\sim}{0}$,

and thus refuted Shoenfield's conjecture.

Later several other inhomogeneities in $\underset{\sim}{R}$ and counter examples to Shoenfield's conjecture were discovered.

Though numerous results about $\underset{\sim}{R}$ have been proved, no characterization of the partial ordering of the r.e. degrees is known, and one seems

to be still far away from an understanding of this structure. Too many detailed problems are still unsolved. One may hope that solving some of these problems may lead to partial decision procedures for the elementary theory of $(\underset{\sim}{R}, \leq)$.

We investigate here several such problems concerning different aspects of the structure of $\underset{\sim}{R}$. Emphasis is laid on questions about noncappable degrees and pairs of r.e. degrees without infimum. We also look, however, at lattice embeddings and at a problem concerning sequences of uniformly r.e. degrees.

In the first four sections we deal with noncappable degrees, i.e. nonzero r.e. degrees which aren't halves of minimal pairs. It is easily seen that the set NC of the noncappable degrees is a filter in $\underset{\sim}{R}$. Since one major aspect of the inhomogeneity of $\underset{\sim}{R}$ is due to minimal pairs, Jockusch conjectured that the set of the noncappable degrees has the homogeneity property of Shoenfield's Conjecture. In section 1 we refute this conjecture by constructing a pair of noncappable degrees which have an infimum (in NC).

For locating NC in $\underset{\sim}{R}$, the following two questions are very important:

(3) Is there strictly below any noncappable degree another noncappable one?

(4) Is there above every incomplete r.e. degree an incomplete noncappable one?

In section 2 we give a partial answer to (3). We introduce effectively noncappable sets and degrees. A degree is effectively noncappable if it contains an effectively noncappable set. The set ENC of the effectively noncappable degrees is a subset of NC and is, as NC, a filter in $\underset{\sim}{R}$. It is not known if ENC is a proper subset of NC. We prove the Sacks- and Robinson-splitting theorems for effectively noncappable sets

and degrees. Obviously both results imply a positive answer to (3) if noncappable is replaced by effectively noncappable. It also follows that ENC - and thus NC - is an automorphism basis for $\underset{\sim}{R}$.

Question (4) is answered positively in section 3.

In section 4 we prove an extension of the non-diamond theorem. This theorem of Lachlan asserts that there is no embedding of the four-element Boolean algebra (diamond) into the upper semi-lattice $(\underset{\sim}{R}, \leq)$ of the r.e. degrees which preserves infima, suprema, and least and greatest elements. I.e. if $\underset{\sim}{a} \cup \underset{\sim}{b} = \underset{\sim}{0}'$ then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ don't form a minimal pair. Jockusch observed that the (slightly modified) proof of this theorem actually shows that if $\underset{\sim}{a} \cup \underset{\sim}{b} = \underset{\sim}{0}'$ then $\underset{\sim}{a}$ or $\underset{\sim}{b}$ is noncappable, and he proved the following extension

(5) $\underset{\sim}{0}' = \underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n \quad \Longrightarrow \quad \exists i \leq n$ ($\underset{\sim}{a}_i$ is noncappable)

(for any $n < \omega$).

Lachlan showed by relativization an extension of the non-diamond theorem in another direction:

(6) ($\underset{\sim}{b}$ low, $\underset{\sim}{b} < \underset{\sim}{a}_0, \underset{\sim}{a}_1$ and $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = 0'$) $\quad \Longrightarrow \quad \neg$ ($\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1 = \underset{\sim}{b}$),

and used this to prove (1).

We prove the following generalization of (6):

(7) ($\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ low and $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$) $\Longrightarrow \exists i \leq 1$ ($\underset{\sim}{a}_i$ is non-$\underset{\sim}{b}_i$-cappable)

where for any r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable if

$\forall \underset{\sim}{c} \neq \underset{\sim}{b} \exists \underset{\sim}{d}$ ($\underset{\sim}{d} \leq \underset{\sim}{a}, \underset{\sim}{c}$ and $\underset{\sim}{d} \neq \underset{\sim}{b}$).

Following Jockusch, we show that n=1 in (7) can be replaced by any $n < \omega$. Further we get the result for an effective notion of non-$\underset{\sim}{b}_i$-cappable. By the latter our extended non-diamond theorem shows that in (5) noncappable may be replaced by effectively noncappable.

The perhaps most interesting consequence of our result is

(8) ($\underset{\sim}{0}' = \underset{\sim}{b} \cup \underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n$ and $\underset{\sim}{b}$ low) $\Longrightarrow$

$\Longrightarrow \quad \exists i \leq n$ ($\underset{\sim}{a}_i$ is (effectively) noncappable

We use (8) to prove (4) for effectively noncappable degrees in place of noncappable degrees. Harrington showed that there exists an incomplete noncappable degree $\underset{\sim}{a}$ such that

$\exists \underset{\sim}{b}$ ($\underset{\sim}{b}$ is half of a minimal pair and $\underset{\sim}{a} \cup \underset{\sim}{b} = \underset{\sim}{0}'$).

(8) implies that not every noncappable degree has that property. Further consequences of the extended non-diamond theorem we prove are the following two results about pairs of r.e. degrees without infima:

(9) $\underset{\sim}{a}$ low $\Longrightarrow$ $\exists \underset{\sim}{b}$ ($\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist)

(10) $\exists \underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}' \forall \underset{\sim}{b}$ ($\underset{\sim}{b}$ low and $\underset{\sim}{a} \mid \underset{\sim}{b}$ $\Longrightarrow$ $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist).

These are partial answers to questions of Jockusch i.e., whether every r.e. degree is half of a pair of r.e. degrees which has no infimum, and whether there exists an r.e. degree $\neq \underset{\sim}{0}, \underset{\sim}{0}'$ which is not half of a pair of incomparable r.e. degrees which has an infimum.

In section 5 we give general positive answers to these questions using a more direct approach. We produce an r.e. degree $\underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}'$ s.t.

$\forall \underset{\sim}{b} < \underset{\sim}{a} \forall \underset{\sim}{c} \not\leq \underset{\sim}{b} \exists \underset{\sim}{d}$ ($\underset{\sim}{d} \leq \underset{\sim}{c}, \underset{\sim}{a}$ and $\underset{\sim}{d} \not\leq \underset{\sim}{b}$).

Such a degree is called strongly noncappable. Since for a strongly noncappable degree $\underset{\sim}{a}$,

$\forall \underset{\sim}{b}$ ($\underset{\sim}{a} \mid \underset{\sim}{b}$ $\Longrightarrow$ $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist),

this answers the second question positively. Further we show that to any r.e. degree $\underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}'$ an incomparable strongly noncappable degree exists, which easily implies an answer to the first question.

Another question about pairs of r.e. degrees without infimum, namely if below any nonzero r.e. degree such a pair exists, could not be answered by the known constructions of pairs without infimum. Using the strongly noncappable degree construction we show

(11) $\forall \underset{\sim}{a} > \underset{\sim}{0} \exists \underset{\sim}{b}_0, \underset{\sim}{b}_1$ ($\underset{\sim}{a} = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ and $\underset{\sim}{b}_0 \cap \underset{\sim}{b}_1$ doesn't exist).

Indeed we can show

(12) $\forall \underset{\sim}{a} > \underset{\sim}{0} \forall \underset{\sim}{c}$ (($\underset{\sim}{c} < \underset{\sim}{a}$ and $\underset{\sim}{c}$ low) $\Longrightarrow$

$\Longrightarrow$ $\exists \underset{\sim}{b}_0, \underset{\sim}{b}_1$ ($\underset{\sim}{a} = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$, $\underset{\sim}{c} < \underset{\sim}{b}_0, \underset{\sim}{b}_1$ and

$\underset{\sim}{b}_0 \cap \underset{\sim}{b}_1$ doesn't exist))

(by Lachlan [6] the assumption that $\underset{\sim}{c}$ is low cannot be dropped).

In sections 6 and 7 we study lattice embeddings into the r.e. degrees which preserve greatest element and greatest and least element, respectively. It is known that every countable distributive lattice is embeddable into $\underset{\sim}{R}$. The situation for nondistributive lattices, however, is still unclear. One only knows that some but not all finite nondistributive lattices are embeddable.

While all the known embeddings automatically preserved the least element, for a longer time it wasn't known if there are nontrivial embeddings which preserve greatest element. Finally Lachlan and independently Shoenfield and Soare proved that the diamond is embeddable into $\underset{\sim}{R}$ by a map which preserves greatest element. Building on Shoenfield and Soare's proof we show that this is true for all countable distributive lattices. The embeddings can be choosen to be into the set ENC of the effectively noncappable degrees. Further we give greatest element preserving embeddings of the five-element non-modular lattice and some more nondistributive lattices of similar structure. By the non-diamond theorem not every finite distributive lattice can be embedded into the r.e. degrees by a map which preserves both greatest and least elements. In section 7 we introduce a subclass of the countable distributive lattices of which all members are embeddable into $\underset{\sim}{R}$ preserving greatest and least element.

In section 8 we look at sequences of uniformly r.e. degrees. Yates proved that there exists an ascending sequence of uniformly r.e. degrees which has an exact pair. It wasn't known, however, if every

such sequence posseses an exact pair. We show that this is not the case. Indeed below any nonzero r.e. degree exists an ascending sequence of uniformly r.e. degrees without an exact pair.

In the final section 9 we turn to one of the generalizations of classical recursion theory, namely α-recursion theory. Instead of subsets of ω, here subsets of admissible ordinals α are studied. (In the special case $\alpha = \omega$, one just gets classical recursion theory.)

Shore proved that for some admissible ordinals α the elementary theory of the partial ordering of the α-r.e. degrees $(\alpha\text{-}\underset{\sim}{R}, \leq)$ is different from that of the partial ordering of the r.e. degrees $(\underset{\sim}{R}, \leq)$. Indeed he provided a Σ_3-sentence which holds in $(\underset{\sim}{R}, \leq)$ but not in $(\alpha\text{-}\underset{\sim}{R}, \leq)$ for some α. We show that differences occur already on the level of the Σ_2-sentences, which is in respect to the Σ_n-hierarchy the best possible result since Lerman proved that in all $(\alpha\text{-}\underset{\sim}{R}, \leq)$ the same Σ_1-sentences are true.

Generalizing (1) to α-recursion theory, we further show that for all admissible α the α-r.e. degrees don't form a lattice.

To prove these results we show that the extended non-diamond theorem (7) holds in α-recursion theory for all admissible ordinals α. It also will follow that incomplete noncappable α-r.e. degrees exist for all α.

Further, for <u>some</u> admissible ordinals α, we get a partition of $\alpha\text{-}\underset{\sim}{R}$ into infinite sets F and I such that F is a filter and I is an ideal and such that

(13) $\forall \underset{\sim}{a} \in F \forall \underset{\sim}{b} \in I \; (\underset{\sim}{a} \mid \underset{\sim}{b} \implies \underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist).

This partition is given by the cuppable and noncuppable α-degrees. In classical recursion theory such a partition is unknown. Here, for the cuppable and noncuppable degrees, not only does (13) fail but

the cuppable r.e. degrees also fail to form a filter as we show.

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§ 0 PRELIMINARIES.

We will deal with subsets of ω, the set of the non-negative integers. Instead of subset of ω, (partial) function from ω^n ($n < \omega$) into ω, and element of ω, we say set, (partial) (n-ary) function, and number. Sets are denoted by capital letters, numbers by lower case letters. The letters f,g and h also denote functions.

$\overline{A}$ is the complement of A, and $A[x]$ the restriction of A to numbers less than x, i.e. $A[x] = \{y : y < x \text{ and } y \in A\}$. We use the same notation for a set and its characteristic function; i.e. $x \in A$ iff $A(x) = 1$, and $x \notin A$ iff $A(x) = 0$. The join $A \oplus B$ of sets A and B is defined by $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$.

If f is a partial function and if $x \in \mathrm{dom}(f)$, we write $f(x)\downarrow$ (i.e. $f(x)$ is defined), otherwise $f(x)\uparrow$ ($f(x)$ undefined). $f(x) = g(y)$ is to read: either $f(x)$ and $g(y)$ are both defined and the same number or both are undefined (To denote only the first case we write $f(x) = g(y) \downarrow$.).

The basic concepts and results of recursion theory which we will need can be found in Rogers [15]. We will use them (in particular that in chapters 5 and 9) without reference.

There exist recursive bijective functions $\lambda x_0, \ldots, x_n (< x_0, \ldots, x_n >)$ from ω^{n+1} onto ω, and recursive functions $\lambda x((x)_i^{n+1}))$ such that $x_i = (< x_0, \ldots, x_n >)_i^{n+1}$ and $< x_0, \ldots, x_n > = < x_0, \ldots, x_{i-1}, < x_i, \ldots, x_n >>$ ($\forall\, i \leq n < \omega$). If no confusion can arise we write $(x)_i$ instead of $(x)_i^{n+1}$. $A^{(x)}$ denotes the set $\{< x, y > : y < \omega \text{ and } < x, y > \in A\}$.
The existence of such recursive coding functions allows us to develop all (recursion theoretic) concepts for only unary func-

tions.

Let $\{\lambda x,X(\{e\}^X(x)) : e < \omega\}$ be a standard enumeration of the Turing reductions (e.g. if the e-th oracle machine with oracle X and input x yields output y, then $\{e\}^X(x) = y$, and if the machine doesn't stop $\{e\}^X(x)\uparrow$).
A (partial) function f is recursive in A (or A-recursive), $f \leq_T A$, if for some $e < \omega$, $f = \{e\}^A$. A set B is recursive in A (A-recursive, $B \leq_T A$) if its characteristic function is recursive in A.

Let $W_e^X = \mathrm{dom}(\{e\}^X)$ (for all $e < \omega$ and $X \subseteq \omega$). A is recursively enumerable in B (r.e. in B, B-r.e.) if $A = W_e^B$ for some $e < \omega$. $\{W_e^X : e < \omega\}$ is a standard enumeration of the X-r.e. sets.

Instead of $\{e\}^\emptyset$ and $W_e^\emptyset$ we write $\{e\}$ and W_e, respectively.
A (partial) function is recursive if $f = \{e\}$ for some $e < \omega$.
A set A is recursive if its characteristic function is recursive, and recursively enumerable (r.e.) if $A = W_e$ for some $e < \omega$.
If $A = W_e$, e is an (r.e.)index of A. $\{W_e : e < \omega\}$ is a standard enumeration of the r.e. sets.

A sequence $\{A_e : e < \omega\}$ is uniformly recursive in a sequence $\{B_e : e < \omega\}$, if there exists a recursive function f s.t. $A_e = \{f(e)\}^B e$. (We also write: $A_e \leq_T B_e$ uniformly in e). If for all $e < \omega$, $A_e = A$ ($B_e = B$), we say that A is uniformly recursive in $\{B_e : e < \omega\}$ ($\{A_e : e < \omega\}$ is uniformly recursive in B), and write $A \leq_T B_e$ uniformly in e ($A_e \leq_T B$ uniformly in e).
Uniformly r.e. in, uniformly recursive and uniformly r.e. are defined analogously.

The canonical index of a set is defined as in Rogers [15], p.70. $\mathbb{D}_x$ denotes the set whose canonical index is x. If $y \in \mathbb{D}_x$, then $y < x$.

A sequence $\{A_s : s < \omega\}$ of finite sets is an effective enumeration of A if $\forall s\,(A_s \subseteq A_{s+1})$, $A = \bigcup\{A_s : s < \omega\}$, and if there exists a recursive function f such that $\forall s\,(A_s = \mathbb{D}_{f(s)})$. A set has an effective enumeration iff it is r.e..

Let $\{\lambda x,X(\{e\}^X_s(x)) : e,s < \omega\}$ be a standard enumeration of approximations to $\{\lambda x,X(\{e\}^X(x) : e < \omega\}$ (e.g. if the e-th oracle machine with oracle X and input x yields output y in less than s steps, then $\{e\}^X_s(x) = y$, otherwise $\{e\}^X_s(x)\uparrow$). These approximations have the following properties (for all $e,x<\omega,X\subseteq\omega$):

(0.1) $\{e\}^X(x)\uparrow \implies \forall s\,(\{e\}^X_s(x)\uparrow)$.

(0.2) $\{e\}^X(x)\downarrow \implies \exists s\,\forall t \geq s\,(\{e\}^X_t(x) = \{e\}^X(x))$.

(0.3) $\forall s\,(\{e\}^X_s(x)\downarrow \implies \forall t \geq s\,(\{e\}^X_t(x) = \{e\}^X(x))$.

Further there exist use-functions $u(e,X,x)$ and $u(e,X,x,s)$ of $\{e\}^X(x)$ and $\{e\}^X_s(x)$, respectively, which have the following properties:

(0.4) $u(e,X,x) = 0$ iff $\{e\}^X(x)\uparrow$, and
$u(e,X,x,s) = 0$ iff $\{e\}^X_s(x)\uparrow$.

(0.5) If $u(e,X,x,s) \neq 0$ and $X[u(e,X,x,s)] = Y[u(e,Y,x,s)]$, then
$\forall t \geq s\,(u(e,X,x,t) = u(e,Y,x,t) = u(e,X,x) = u(e,Y,x)$
and
$\{e\}^X_t(x) = \{e\}^Y_t(x) = \{e\}^X(x) = \{e\}^Y(x)\downarrow)$.

((0.5) is called use-principle.)

Without loss of generality we may assume that for all $e,s,x < \omega$ and all $X \subseteq \omega$ the following is true:

(0.6) $\{e\}^X_s(x)\downarrow \implies e,x,u(e,X,x,s) < s.$

This and the use-principle imply

(0.7) $\{e\}^X_s(x)\downarrow$ and $X[s] = Y[s] \implies$
$\implies \forall t \geq s\ (\{e\}^X_t(x) = \{e\}^Y_t(x) = \{e\}^X(x) = \{e\}^Y(x)\downarrow)$.

A computation $\{e\}^A_s(x)$ which is defined is <u>B-correct</u> if $A[u(e,A,x,s)] \supseteq B[u(e,A,x,s)]$.

If A is an r.e. set and $\{A_s : s < \omega\}$ an effective enumeration of A, then

(0.8) $\lambda e,x,s(\{e\}^{A_s}_s(x))$ is a partial recursive function whose domain is recursive.

(0.9) $\lambda e,x,s(u(e,A_s,x,s))$ is a recursive function.

(0.10) $\{e\}^A(x)\downarrow \implies \exists s\,\forall t \geq s\ (\{e\}^{A_t}_t(x) = \{e\}^A(x)$ and $u(e,A_t,x,t) = u(e,A,x))$.

(0.11) $\{e\}^{A_s}_s(x)$ A-correct $\implies$
$\implies \forall t \geq s\ (\{e\}^{A_t}_t(x) = \{e\}^A(x)$ and $u(e,A_t,x,t) = u(e,A,x))$.

We set $W^X_{e,s} = \mathrm{dom}(\{e\}^X_s)$, $\{e\}_s = \{e\}^\emptyset_s$, and $W_{e,s} = W^\emptyset_{e,s}$. $\{W_{e,s} : s < \omega\}$ is an effective enumeration of W_e. Set $K = \{\langle x,y\rangle : y \in W_x\}$ and $K_s = \{\langle x,y\rangle : y \in W_{x,s}\}$. Then K is a <u>complete</u> set, i.e. K is r.e. and every r.e. set is recursive in K, and $\{K_s : s < \omega\}$ is an effective enumeration of K.

If A is not recursive in B, we write $A \not\leq_T B$. If $A \leq_T B$ and $B \not\leq_T A$, then $A <_T B$. If $A \leq_T B$ and $B \leq_T A$, then $A \equiv_T B$, and otherwise $A \not\leq_T B$. If $A \not\leq_T B$ and $B \not\leq_T A$, then we say that A and B are <u>T-incomparable</u>, and write $A \mid_T B$.

The relation $\leq_T$ (restricted to sets) is reflexive and transitive.

Hence $\equiv_T$ is an equivalence relation. The equivalence classes are called degrees. We write dg(A) for the degree of A, i.e. for the set $\{B : B \equiv_T A\}$. If a degree contains an r.e. set, the degree is recursively enumerable (r.e.). R.e. degrees are denoted by $\underset{\sim}{a}$, $\underset{\sim}{b}$, etc. and $\underset{\sim}{R}$ denotes the set of the r.e. degrees. A partial ordering on $\underset{\sim}{R}$ is induced by $\leq_T$:

$\underset{\sim}{a} \leq \underset{\sim}{b}$ if $\exists A \in \underset{\sim}{a}, \exists B \in \underset{\sim}{b}$ ($A \leq_T B$).

We write

$\underset{\sim}{a} < \underset{\sim}{b}$ if $\underset{\sim}{a} \leq \underset{\sim}{b}$ and $\underset{\sim}{a} \neq \underset{\sim}{b}$,

$\underset{\sim}{a} \not< \underset{\sim}{b}$ if not $\underset{\sim}{a} \leq \underset{\sim}{b}$,

$\underset{\sim}{a} \mid \underset{\sim}{b}$ if $\underset{\sim}{a} \not< \underset{\sim}{b}$ and $\underset{\sim}{b} \not< \underset{\sim}{a}$, and

$\underset{\sim}{a} \nmid \underset{\sim}{b}$ if not $\underset{\sim}{a} \mid \underset{\sim}{b}$.

If $\underset{\sim}{a} \mid \underset{\sim}{b}$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are incomparable.

There exists (in respect to $\leq$) a least r.e. degree $\underset{\sim}{0}$, which contains just the recursive sets, and a greatest degree $\underset{\sim}{0}'$, which contains the complete sets.

$\underset{\sim}{a} \cup \underset{\sim}{b}$ denotes the supremum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, i.e. $\underset{\sim}{c} = \underset{\sim}{a} \cup \underset{\sim}{b}$ if $\underset{\sim}{a}, \underset{\sim}{b} \leq \underset{\sim}{c}$ and $\forall \underset{\sim}{d}$ ($\underset{\sim}{a}, \underset{\sim}{b} \leq \underset{\sim}{d} \implies \underset{\sim}{c} \leq \underset{\sim}{d}$). [1]

For all r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{a} \cup \underset{\sim}{b}$ exists. If $A \in \underset{\sim}{a}$ and $B \in \underset{\sim}{b}$ then $\underset{\sim}{a} \cup \underset{\sim}{b} = dg(A \oplus B)$.

In contrast to the supremum, the infimum $\underset{\sim}{a} \cap \underset{\sim}{b}$ of two degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ doesn't exist for some r.e. degrees.

Hence $\underset{\sim}{R}(\leq, \cup, \cap)$ is an upper-semilattice but no lattice.

$\underset{\sim}{a} \cap \underset{\sim}{b} = \underset{\sim}{c}$ is to read that $\underset{\sim}{a} \cap \underset{\sim}{b}$ exists and is equal to $\underset{\sim}{c}$. Hence $\underset{\sim}{a} \cap \underset{\sim}{b} \neq \underset{\sim}{c}$ if the infimum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ doesn't exist or if it exists

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1) We use $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_n \leq \underset{\sim}{b}_0, \ldots, \underset{\sim}{b}_m$ as an abbreviation of $\forall i \leq n \, \forall j \leq m$ ($\underset{\sim}{a}_i \leq \underset{\sim}{b}_j$). (Analogously for $=$, $<$, etc. in place of $\leq$).

and is different from $\underset{\sim}{c}$.

The jump A′ of a set A is the set $\{\langle x,y\rangle : y \in W_x^A\}$. Properties of the jump are: $A \leq_T A'$, A' r.e. in A, $A' \not\leq_T A$, and if $A \leq_T B$ then $A' \leq_T B'$. By the last property the jump $\underset{\sim}{a}' = dg(A')$ of a degree $\underset{\sim}{a} = dg(A)$ is welldefined. For all r.e. degrees $\underset{\sim}{a}$, $\underset{\sim}{a} < \underset{\sim}{a}'$, $\underset{\sim}{0}' \leq \underset{\sim}{a}'$ and if $\underset{\sim}{a} \neq \underset{\sim}{b}$ then $\underset{\sim}{a}' \neq \underset{\sim}{b}'$. (Note that the definition of the jump is compatible with the earlier definition of $\underset{\sim}{0}$ and $\underset{\sim}{0}'$.)
A set A is called low if $A' \equiv_T \emptyset'$. The degree $\underset{\sim}{a}$ is low if it contains (only) low sets, i.e. if $\underset{\sim}{a}' = \underset{\sim}{0}'$.

An automorphism of $\underset{\sim}{R}$ is a bijective map $f : \underset{\sim}{R} \longrightarrow \underset{\sim}{R}$ s.t.

$$\forall \underset{\sim}{a},\underset{\sim}{b}\ (\ f(\underset{\sim}{a} \cup \underset{\sim}{b}) = f(\underset{\sim}{a}) \cup f(\underset{\sim}{b})\ )$$

and

$$\forall \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{a} \cap \underset{\sim}{b} \text{ exists } \Longrightarrow f(\underset{\sim}{a} \cap \underset{\sim}{b}) = f(\underset{\sim}{a}) \cap f(\underset{\sim}{b})\ ).$$

A subset $\underset{\sim}{A}$ of $\underset{\sim}{R}$ is an automorphism-basis if for every automorphism f

$$\exists \underset{\sim}{a}\ (\ f(\underset{\sim}{a}) \neq \underset{\sim}{a}\ ) \Longrightarrow \exists \underset{\sim}{a} \in \underset{\sim}{A}\ (\ f(\underset{\sim}{a}) \neq \underset{\sim}{a}\ ).$$

By the Sacks splitting theorem, the low degrees are an automorphism-basis.

Two classical results on r.e. degrees (by Sacks), which we will often use, are:

Splitting theorem. For all r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ s.t. $\underset{\sim}{b} > \underset{\sim}{0}$, there exist low degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$.

Density theorem. If $\underset{\sim}{a} < \underset{\sim}{b}$ then there is a $\underset{\sim}{c}$ s.t. $\underset{\sim}{a} < \underset{\sim}{c} < \underset{\sim}{b}$.

(Proofs of these theorems can be found in Soare [26].)

Most results on the r.e. degrees are proved by construction of r.e. sets s.t. their degrees have the desired properties. The

basic method to do this is the priority method developed by Friedberg [2] and Muchnik [13]: A set A (or several sets) is constructed in ω many stages. The construction will be effective, and at any stage only finitely many numbers are put into A. Hence, if A_s is the set of numbers put into A by the end of stage s, $\{A_s : s < \omega\}$ will be an effective enumeration of A and thus A will be r.e.. To guarantee that A has the desired properties, an infinite sequence $\{R_n : n < \omega\}$ of conditions, called requirements, will be satisfied by A. There are two types of requirements, positive and negative ones. To meet a positive requirement we have to put certain numbers into A, to meet a negative requirement we have to keep a certain finite set of numbers out of A. Hence there can be a conflict between a positive requirement R_n and a negative requirement R_m : To meet R_n we want to put x into A, while to meet R_m we want to keep x out of A. The conflict is solved by giving the requirement with smaller index priority over the other. Hence if $n < m$, we put x into A thus satisfying R_n and injuring R_m. We have to show that eventually all requirements are satisfied.

In the following constructions, if the set A (A_i ($i < \omega$), B etc.) is constructed, A_s ($A_{i,s}$, B_s etc.) denotes the finite set of numbers put into A (A_i , B etc.) by the end of stage s ($s < \omega$).

Finally, we should mention that we don't use a special formalization of the notions "recursive", "r.e." etc. (as e.g. given by Turing machines and μ-recursive functions) but use these concepts in an intuitive way. It isn't difficult, however, (though rather tedious) to formalize the given proofs.

§ 1 ON A CONJECTURE OF JOCKUSCH ON NONCAPPABLE DEGREES.

In [18] Shoenfield stated the wellknown conjecture that the set of the r.e. degrees has a certain homogeneity property (for an explanation see Rogers [15], p 168/9). The conjecture, if correct, had yielded a complete characterization of the structure of the r.e. degrees and a decision procedure for the elementary theory of $(\underset{\sim}{R},\leq)$. Some main consequences of the conjecture (as listed by Shoenfield) are : For all r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(1.1) $\underset{\sim}{a}<\underset{\sim}{b} \Rightarrow \exists \underset{\sim}{c}\ (\ \underset{\sim}{a}<\underset{\sim}{c}<\underset{\sim}{b}\ )$,

(1.2) $\underset{\sim}{0}<\underset{\sim}{a}<\underset{\sim}{b} \Rightarrow \exists \underset{\sim}{c}\ (\ \underset{\sim}{c}<\underset{\sim}{b}$ and $\underset{\sim}{a}\cup\underset{\sim}{c}=\underset{\sim}{b}\ )$, and

(1.3) $\underset{\sim}{a} \mid \underset{\sim}{b} \Rightarrow \underset{\sim}{a}\cap\underset{\sim}{b}$ doesn't exist.

(1.1) was proved by Sacks (Density Theorem). Lachlan and Yates, however, disproved (1.3) and thus refuted Shoenfield's conjecture. Both showed

(1.4) $\exists \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{a} \mid \underset{\sim}{b}$ and $\underset{\sim}{a}\cap\underset{\sim}{b}=\underset{\sim}{0}\ )$.

Lachlan also proved that there are incomparable r.e. degrees which have an infimum different from $\underset{\sim}{0}$. Later Lachlan disproved (1.2) too, and Yates and Cooper got the stronger result

(1.5) $\exists \underset{\sim}{a}>\underset{\sim}{0}\ \forall \underset{\sim}{c}<\underset{\sim}{0}'\ (\ \underset{\sim}{a}\cup\underset{\sim}{c}<\underset{\sim}{0}')$,

which says that (1.2) fails for $\underset{\sim}{b}=\underset{\sim}{0}'$. Further counterexamples to (1.2) were found by Harrington.

Yates proved that for some r.e. degrees $\underset{\sim}{a}(\neq\underset{\sim}{0}')$ no $\underset{\sim}{b}>\underset{\sim}{0}$ exists such that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ witness (1.4). Such degrees $\underset{\sim}{a}$ are called noncappable (or simple). Since counterexample (1.4) is due to degrees which aren't noncappable, Jockusch conjectured (cf. Soare [28], p 1174) that the original Shoenfield conjecture holds for the noncappable degrees together with $\underset{\sim}{0}$. This conjecture was further supported by

results of Harrington and Jockusch himself which imply that also the counterexample (1.5) fails for the noncappable degrees.
We show, however, that Jockusch's conjecture is not true..There exist two incomparable noncappable degrees which have an infimum in the set of the noncappable degrees. I.e. (1.3) for noncappable degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and thus the general conjecture fail.
In this section we give our original (direct) proof of this result. An embedding result we prove in §6 will also imply that there are incomparable noncappable degrees which have an infimum (cf. Corollary 6.3).

We begin with the basic definitions and some easy facts on noncappable degrees.

Definition 1.1. (a) R.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are a minimal pair if $\underset{\sim}{a} \mid \underset{\sim}{b}$ and $\underset{\sim}{a} \cap \underset{\sim}{b} = \underset{\sim}{0}$.

(b) $\underset{\sim}{a}$ is cappable if, for some $\underset{\sim}{b} > \underset{\sim}{0}$, $\underset{\sim}{a} \cap \underset{\sim}{b} = \underset{\sim}{0}$. Otherwise $\underset{\sim}{a}$ is noncappable.

NC denotes the set of the noncappable degrees and $NC_0 = NC \cup \{\underset{\sim}{0}\}$.

In the literature noncappable degrees are also called simple. Since confusion with simple sets may arise, we use the more recent notation "noncappable".

Note that $\underset{\sim}{0}$ is cappable and $\underset{\sim}{0}'$ is noncappable. A nonzero degree is cappable iff it is half of a minimal pair.

Proposition 1.1. The set NC of the noncappable degrees is a filter in $\underset{\sim}{R}$. I.e for all r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

(a) If $\underset{\sim}{a} \in NC$ and $\underset{\sim}{a} < \underset{\sim}{b}$ then $\underset{\sim}{b} \in NC$.

(b) If $\underset{\sim}{a}, \underset{\sim}{b} \in NC$ and $\underset{\sim}{a} \cap \underset{\sim}{b}$ exists then $\underset{\sim}{a} \cap \underset{\sim}{b} \in NC$.

Proof. (a) : Fix $\underset{\sim}{a}$ and $\underset{\sim}{b}$ s.t. $\underset{\sim}{a} \in NC$ and $\underset{\sim}{a} < \underset{\sim}{b}$. Assume for a contradiction that $\underset{\sim}{b}$ is cappable and choose $\underset{\sim}{c} > \underset{\sim}{0}$ such that $\underset{\sim}{b} \cap \underset{\sim}{c} = \underset{\sim}{0}$. Then, since $\underset{\sim}{a} < \underset{\sim}{b}$, $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$ and thus $\underset{\sim}{a}$ is cappable. Contradiction.

(b) : Fix noncappable degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ s.t. $\underset{\sim}{a} \cap \underset{\sim}{b}$ exists, and let $\underset{\sim}{c}$ be any nonzero r.e. degree. We have to show that there exists an $\underset{\sim}{e} > \underset{\sim}{0}$ s.t. $\underset{\sim}{e} \leq \underset{\sim}{a} \cap \underset{\sim}{b}, \underset{\sim}{c}$.

Since $\underset{\sim}{a}$ is noncappable, there exists $\underset{\sim}{d} > \underset{\sim}{0}$ s.t. $\underset{\sim}{d} \leq \underset{\sim}{a}, \underset{\sim}{c}$. Then, since $\underset{\sim}{b} \in NC$, there exists $\underset{\sim}{e} > \underset{\sim}{0}$ s.t. $\underset{\sim}{e} \leq \underset{\sim}{b}, \underset{\sim}{d}$. It follows that $\underset{\sim}{e} \leq \underset{\sim}{a} \cap \underset{\sim}{b}, \underset{\sim}{c}$.

Only a different formulation of proposition 1.1 (a) is

(1.6) If $\underset{\sim}{a}$ is cappable and $\underset{\sim}{b} < \underset{\sim}{a}$ then $\underset{\sim}{b}$ is cappable.

Basic results on the existence of cappable and noncappable degrees are :

(1.7) (Lachlan [4] and Yates [33]) There exists a minimal pair of r.e. degrees, i.e. there exists a nonzero cappable degree.

(1.8) (Yates [33]) There exists an incomplete noncappable degree.

An easy consequence of (1.7) and (1.6) is

(1.9) Below any nonzero r.e. degree is a nonzero cappable degree.

(Proof : Fix $\underset{\sim}{a} > \underset{\sim}{0}$ to show that there is a cappable $\underset{\sim}{b}$ s.t. $\underset{\sim}{0} < \underset{\sim}{b} \leq \underset{\sim}{a}$. W.l.o.g. $\underset{\sim}{a} \in NC$. By (1.7) choose a cappable degree $\underset{\sim}{c} > \underset{\sim}{0}$. Since $\underset{\sim}{a}$ is noncappable, there exists $\underset{\sim}{b}$ s.t $\underset{\sim}{0} < \underset{\sim}{b} \leq \underset{\sim}{a}, \underset{\sim}{c}$. By (1.6) $\underset{\sim}{b}$ is cappable.)

For the proof of the main result of this section we need the following definition.

Definition 1.2. An r.e. degree $\underset{\sim}{a}$ is branching if there are r.e. degrees $\underset{\sim}{b}$, $\underset{\sim}{c}$ both $> \underset{\sim}{a}$ and such that $\underset{\sim}{a} = \underset{\sim}{b} \cap \underset{\sim}{c}$. Otherwise $\underset{\sim}{a}$ is nonbranching.

(1.7) says that $\underset{\sim}{0}$ is branching. Lachlan proved that there are nonzero branching degrees. We prove the following extension of this result.

Theorem 1.1. There exists a branching noncappable degree.

Comllary 1.1. The Jockusch conjecture on noncappable degrees fails.

Proof. By the theorem choose r.e. degrees $\underset{\sim}{a},\underset{\sim}{b}$ and $\underset{\sim}{c}$ s.t. $\underset{\sim}{a}\in NC$, $\underset{\sim}{a}<\underset{\sim}{b},\underset{\sim}{c}$, and $\underset{\sim}{a}=\underset{\sim}{b}\cap\underset{\sim}{c}$. Then $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are incomparable degrees which, by proposition 1.1 (a), are noncappable and which have infimum $\underset{\sim}{a}$ in the set of the noncappable degrees. I.e. (1.3) fails for noncappable degrees, and thus the Jockusch conjecture fails.

Proof of theorem 1.1. The proof combines the Yates noncappable degree construction in [33] with the Lachlan construction of a nonzero branching degree in [4]. We will use a simplified version of the latter construction by Soare, as described in [30], Chapter IX.

We construct pairwise disjoint r.e. sets A, B_0, and B_1 such that, for $\underset{\sim}{a}=dg(A)$, $\underset{\sim}{b}_0=dg(A\cup B_0)$, and $\underset{\sim}{b}_1=dg(A\cup B_1)$, $\underset{\sim}{a}$ is noncappable,

(1.10) $\underset{\sim}{a}<\underset{\sim}{b}_0,\underset{\sim}{b}_1$, and

(1.11) $\underset{\sim}{a}=\underset{\sim}{b}_0\cap\underset{\sim}{b}_1$.

Since, for $i\leq 1$, $A\leq_T A\cup B_i$ and $B_i\leq_T A\cup B_i$, (1.10) will follow if we insure $B_i\not\leq_T A$. This is done by meeting the requirements

P_{2e+i} : $B_i\neq\{e\}^A$

for all $e<\omega$ and $i\leq 1$.

To get (1.11), we have to show that every function which is recursive in both $A\cup B_0$ and $A\cup B_1$ is recursive in A. This is the case if, for all $e,e_0,e_1<\omega$, where $e=\langle e_0,e_1\rangle$, the following requirements are met :

N_e' : $\{e_0\}^{A\cup B_0} = \{e_1\}^{A\cup B_1}$ total $\Rightarrow$ $\{e_0\}^{A\cup B_0}$ recursive in A.

As Posner observed (cf. Soare [30], Chapter IX), to satisfy all N_e', $e<\omega$, it suffices to satisfy for every $e<\omega$ the requirement

N_e : $\{e\}^{A\cup B_0} = \{e\}^{A\cup B_1}$ total $\Rightarrow$ $\{e\}^{A\cup B_0}$ recursive in A.

(Proof : W.l.o.g. $B_0 \not\subseteq B_1$, say $x_0 \in B_0 \setminus B_1$. For every $e=\langle e_0,e_1\rangle$ there exists an index e' s.t.

$$\{e'\}^X(x) = \begin{cases} \{e_0\}^X(x) & \text{if } x_0 \in X \\ \{e_1\}^X(x) & \text{otherwise} \end{cases}$$

for all $x<\omega$ and $X\subseteq\omega$. Obviously, if $N_{e'}$ is met then N_e' is met too.)

To make $\underset{\sim}{a}=dg(A)$ noncappable, we have to find for every index e an r.e. set A_e such that $A_e \leqq_T A, W_e$ and

(1.12) W_e nonrecursive $\Rightarrow A_e$ nonrecursive.

We let $A^{(e+1)}$ be this set A_e. Then automatically, $A^{(e+1)} \leqq_T A$. $A^{(e+1)} \leqq_T W_e$ is achieved by permitting. In the following construction of A a number x is enumerated in $A^{(e+1)}$ at a stage s only if there is a y such that $y<x$ and $y \in W_{e,s+1} \setminus W_{e,s}$.

(1.12) follows if we meet the requirements

S_e : W_{e_0} nonrecursive $\Rightarrow$ $A^{(e_0+1)} \neq \{e_1\}$

for all $e<\omega$, $e=\langle e_0,e_1\rangle$.

The priority ordering of the requirements will be :

$$N_0 < P_0 < P_1 < S_0 < \cdots \quad N_{n+1} < P_{2(n+1)} < P_{2(n+1)+1} < S_{n+1} < \cdots$$

The negative requirements N_e are handled as in Soare [30]. A length-function l is defined by

$$l(e,s) = \max\{ x : \forall y<x \ (\{e\}_s^{A_s\cup B_0,s}(y) = \{e\}_s^{A_s\cup B_1,s}(y)\downarrow).$$

A restraint-function r and e-maximal stages are defined by simultaneous induction on e, $e<\omega$:

s is <u>0-maximal</u> if $\forall t<s \ (l(0,t)<l(0,s))$,

$$r(0,s) = \begin{cases} 0 & \text{if } s \text{ is 0-maximal} \\ \text{greatest 0-maximal stage } t<s & \text{otherwise,} \end{cases}$$

s is <u>(e+1)-maximal</u> if

$$\forall t<s \ (\ r(e,t) = r(e,s) \implies l(e+1,t) < l(e+1,s) \ ).$$

If s is (e+1)-maximal,

$$r(e+1,s) = \max (\{ r(e,s) \} \cup \{ t<s : r(e,t) < r(e,s) \} \ ),$$

otherwise

$$r(e+1,s) = \max (\{ r(e,s) \} \cup \{ t<s : r(e,t) < r(e,s) \} \cup \{ t<s : r(e,t) = r(e,s) \text{ and } t \text{ (e+1)-maximal} \} \ ).$$

For the requirements P_{2e+i} and S_e followers are appointed. A follower of P_{2e+i} is intended to witness that the requirement is met. If x is a follower of P_{2e+i} at stage s and $\{e\}^A_s(x) = 0$ then we want to put x into B_i and to put a restraint on A to protect the computation $\{e\}^A_s(x)$.

The role of followers of requirement S_e, $e=\langle e_0,e_1\rangle$, is slightly different. Again we look for witnesses that S_e is met , i.e. for numbers y such that $A^{(e_0+1)}(y) \neq \{e_1\}(y)$. In this case, however, only numbers y which are permitted by W_{e_0} will be allowed to enter $A^{(e_0+1)}$. This requires us to be more liberal. Followers of S_e are only lower bounds for potential witnesses. To meet the negative requirements it is only necessary that together with a witness y which is put into $A^{(e_0+1)}$ a follower $x<y$ is put into A.

The requirement P_{2e+i}, $e<\omega$, $i \leq 1$, <u>requires attention at stage</u> s+1 if no (uncancelled) follower of P_{2e+i} is in $B_{i,s}$ and one of the following conditions holds :

(1.13) P_{2e+i} has no follower at the end of stage s.

(1.14) For some follower x of P_{2e+i} (at the end of stage s) $\{e\}^A_s(x) = 0$ and $r(e,s) \leq x$.

(1.15) (1.13) and (1.14) fail and for all followers x of P_{2e+i} (at the end of stage s) $\{e\}^{A}_{s}(x) = 0$.

If P_{2e+i} requires attention at stage s+1 and (1.13) ((1.14),(1.15)) holds then P_{2e+i} <u>requires attention via</u> (1.13) ((1.14),(1.15)) at stage s+1.

The requirement S_e, $e < \omega$, $e = \langle e_0, e_1 \rangle$, <u>requires attention at stage</u> s+1 if

(1.16) $\forall x \, (\{e_1\}_s(\langle e_0+1, e, x \rangle) \downarrow \implies \implies A_s(\langle e_0+1, e, x \rangle) = \{e_1\}_s(\langle e_0+1, e, x \rangle))$,

and one of the following conditions holds :

(1.17) At the end of stage s S_e has no follower.

(1.18) There exist numbers x and y such that

(1.18.1)
$$\begin{cases} x \text{ is a follower of } S_e \text{ at the end of stage } s, \\ x < y, \\ \text{for some } y' < \omega \, , \; y = \langle e_0+1, e, y' \rangle, \\ \{e_1\}_s(y) = 0 \text{ , and} \\ \exists z < y \; (\; z \in W_{e_0,s+1} \setminus W_{e_0,s} \;) \end{cases}$$

If S_e requires attention at stage s+1 and (1.17) ((1.18)) holds then S_e <u>requires attention via</u> (1.17) ((1.18)) at stage s+1.

We can now describe the <u>construction</u> of A, B_0 and B_1 :

<u>Stage 0</u> : Do nothing.

<u>Stage s+1</u> : Let R be the highest priority requirement which requires attention at stage s+1. Cancel all followers of lower priority requirements.

<u>Case 1</u> : $R = P_{2e+i}$, $i \leq 1$. If P_{2e+i} requires attention via (1.13) or (1.15), appoint $\langle 0, s+1 \rangle$ as P_{2e+i}- follower. Otherwise let x be the least follower of P_{2e+i} such that $\{e\}^{A}_{s}(x) = 0$ and

$x \geq r(e,s)$. Put x into B_1.

Case 2 : $R = S_e$, $e = \langle e_0, e_1 \rangle$. If S_e requires attention via (1.17), appoint $\langle 0, s+1 \rangle$ as S_e - follower. Otherwise choose $\langle x,y \rangle$ minimal such that x and y satisfy (1.18.1). Put x and y into A.

We say that R receives attention at stage s+1 or that R is active at stage s+1.

Notice that the construction is effective, and thus the sets A, B_0, and B_1 are r.e.. Obviously they are pairwise disjoint. Further notice that for followers x and y of different requirements $x < y$ iff the requirement x follows has higher priority than that of y. To show that the constructed sets have the desired properties, we prove the following lemmata.

Lemma 1.1. For all $e < \omega$, $\lim\inf_s r(e,s) < \omega$.

Proof by induction on e. If there are infinitely many 0-maximal stages, then $\lim\inf_s r(0,s) = 0$. Otherwise $\lim\inf_s r(0,s)$ is the greatest 0-maximal stage. For the inductive step fix e and assume by induction hypothesis that $k = \lim\inf_s r(e,s)$. Then for at most finitely many stages s $r(e,s) < k$. Let t be the largest such or $t = 0$ if no such stage exists. Let $S = \{ s : r(e,s) = k \}$. If there are infinitely many (e+1)-maximal stages in S, then $\lim\inf_s r(e+1,s) = \max \{ t,k \}$. Otherwise $\lim\inf_s r(e+1,s) = \{ t,k,v \}$, where v is the largest (e+1)-maximal stage in S.

Let $r(e) = \lim\inf_s r(e,s)$ and $S_e = \{ s : r(e,s) = r(e) \}$.

Lemma 1.2. For all $e < \omega$, $i \leq 1$, P_{2e+i} and S_e require attention only finitely often.

Proof by induction on the priority rank of the requirement. For

the inductive step fix a requirement R.

Case 1 : $R = P_{2e+i}$, $i \leq 1$. By induction hypothesis fix a stage s_0 after which no higher priority requirement requires attention. After stage s_0 P_{2e+i} receives attention whenever it requires attention, and no follower of P_{2e+i} is cancelled after this stage. Hence P_{2e+i} requires at most once attention via (1.14) after s_0. Now assume for a contradiction that P_{2e+i} requires attention infinitely often. Then at infinitely many stages followers of P_{2e+i} are appointed. By lemma 1.1 choose a stage $s_1 > \max\{s_0, r(e)\}$ where $\langle 0, s_1 \rangle$ is appointed as follower of P_{2e+i}, and a stage $s_2+1 > s_1$ at which P_{2e+i} requires attention. Then

$$(+) \qquad \{e\}^{A}_{s_2}(\langle 0, s_1 \rangle) = \{e\}^{A}_{s_2+1}(\langle 0, s_1 \rangle) = 0,$$

and all followers of lower priority requirements are cancelled at stage s_2+1. Hence, by choice of s_0, no number less than s_2+1 is enumerated in A after stage s_2. It follows that the computations in (+) are A-correct, and hence

$$\forall s \geq s_2 (\{e\}^{A}_{s}(\langle 0, s_1 \rangle) = 0).$$

Now fix $s_3 > s_2$ such that $r(e, s_3) = r(e)$. Then either some follower of P_{2e+i} is in B_i at the end of stage s_3 or P_{2e+i} requires attention via (1.14) at stage s_3+1, and x or a smaller follower of P_{2e+i} is enumerated in B_i at stage s_3+1. Since no followers of P_{2e+i} are cancelled after stage $s_0 < s_3$, in either case P_{2e+i} never requires attention after stage s_3+1. Contradiction.

Case 2 : $R = S_e$, $e = \langle e_0, e_1 \rangle$. By induction hypothesis choose a stage s_0 after which no requirement of higher priority than S_e requires attention. Then S_e requires attention at most once via (1.17) and once via (1.18) after stage s_0.

Lemma 1.3. For all $e < \omega, i \leq 1$, P_{2e+i} is met.

Proof. Fix $e<\omega$ and $i \leq 1$. First assume that there is a never cancelled follower x of P_{2e+i} in B_i. Choose the stage s_0+1 at which x is enumerated in B_i. Then $\{e\}^{A}_{s_0}(x) = 0$, no number enters A at stage s_0+1, and all followers of lower priority requirements are cancelled at stage s_0+1. Since x is never cancelled, no higher priority requirement is active after stage s_0. It follows that only numbers $> s_0+1$ are put into A after stage s_0, and thus the computation $\{e\}^{A}_{s_0}(x)$ is A-correct. Hence $B_i(x)=1\neq 0=\{e\}^{A}(x)$, i.e. P_{2e+i} is met.

Now assume that no never cancelled follower of P_{2e+i} is in B_i. Further assume for contradiction that P_{2e+i} is not met. By lemma 1.2 fix a stage s_1 after which neither P_{2e+i} nor a higher priority requirement requires attention. Let $x_0,\ldots x_n$, $n \geq -1$, be the followers of P_{2e+i} at the end of stage s_1. By choice of s_1 none of these followers is cancelled. Hence by assumptions, for all $k \leq n$, $B_i(x_k) = \{e\}^{A}(x_k) = 0$. Choose the least stage s such that $s > s_1$ and $\forall k \leq n$ ($B_{i,s}(x_k) = \{e\}^{A}_{s}(x_k) = 0$). Then P_{2e+i} requires attention via (1.13) (if $k = -1$) or via (1.14) or (1.15) at stage s contrary to the choice of $s_1 < s$. This completes the proof of lemma 1.3.

Lemma 1.4. For all $e < \omega$, S_e is met.

Proof. Fix $e<\omega$, $e=\langle e_0,e_1\rangle$, and choose s_0 as in the second case of the proof of lemma 1.2. Then S_e has a follower, say x, at the end of stage s_0+1. Now assume that $A^{(e_0+1)} = \{e_1\}$. We have to show that W_{e_0} is recursive.

A number $\langle e_0+1,e,y\rangle$, $y<\omega$, is enumerated in A only if $\{e_1\}(\langle e_0+1,e,y\rangle) = 0$. Hence $A^{(e_0+1)} = \{e_1\}$ implies

$$\forall y \; (\; A^{(e_0+1)}(\langle e_0+1,e,y\rangle) = \{e_1\}(\langle e_0+1,e,y\rangle) = 0 \;).$$

It follows that (1.16) is true for all stages s. Let $\lambda y(s(y))$ be the recursive function defined by

$$s(y) = \mu s \ (\ s \geq s_0+1 \text{ and } \{e_1\}_s(\langle e_0+1, e, y\rangle) = 0 \ ).$$

Now if, for some $y > x$ and $s \geq s(y)$, $W_{e_0,s+1}[y] \neq W_{e_0,s}[y]$ then S_e requires and receives attention via (1.18) at stage s+1, x is put into $A^{(e_0+1)}$, and hence $A^{(e_0+1)} \neq \{e_1\}$. Since the latter contradicts our assumption, it follows, for all $y > x$, that

$$W_{e_0,s(y)}[y] = W_{e_0}[y],$$

and hence that W_{e_0} is recursive.

Lemma 1.5. For all $e < \omega$, $A^{(e+1)} \leq_T W_e$.

Proof. By construction a number x is enumerated in $A^{(e+1)}$ at stage s+1 only if $W_{e,s}[x] \neq W_e[x]$. Hence to decide recursively in W_e if any given number x is in $A^{(e+1)}$, find an s such that $W_{e,s}[x] = W_e[x]$. Then $x \in A^{(e+1)}$ iff $x \in A_s^{(e+1)}$.

Lemma 1.6. For all $e < \omega$, N_e is met.

Proof. Fix e and, by lemma 1.2, choose s_0 after which no requirement of higher priority than N_e requires attention and such that

$$\forall s \geq s_0 \ (\ r(e-1,s) \geq r(e-1) \ ).$$

Now assume that $\{e\}^{A \cup B_0} = \{e\}^{A \cup B_1} = f$ is a total function. Then $\lim_s l(e,s) = \omega$ and S_{e-1} contains infinitely many e-maximal stages. To A-recursively compute f(p), for given p, find an e-maximal stage $s_1 \in S_{e-1}$ s.t. $s_1 \geq s_0$, $l(e,s_1) > p$, and at least one of the computations $\{e\}_{s_1}^{(A \cup B_0)_{s_1}}(p)$ and $\{e\}_{s_1}^{(A \cup B_1)_{s_1}}(p)$ is A-correct. We claim that $f(p) = q$, where

$$q = \{e\}_{s_1}^{(A \cup B_0)_{s_1}}(p) = \{e\}_{s_1}^{(A \cup B_1)_{s_1}}(p)$$

(and hence $f \leq_T A$).

To prove this, it suffices to show that for all $s \geq s_1$

(1.19) If $s \in S_{e-1}$ and s is e-maximal then

$$\{e\}_s^{(A \cup B_0)_s}(p) = \{e\}_s^{(A \cup B_1)_s}(p) = q$$

and one of the computations is A-correct.

(1.19) is proved by induction on $s \geq s_1$. For $s = s_1$, (1.19) holds by choice of s_1. For the inductive step fix $s > s_1$ such that $s \in S_{e-1}$ and s is e-maximal. Then $l(e,s) > p$. Hence it suffices to show

(1.20) $\exists i \leq 1$ ($\{e\}_s^{(A \cup B_i)_s}(p) = q$ and the computation is A-correct).

Let t be the largest e-maximal stage $< s$ in S_{e-1}. By choice of s_1, $s_1 \leq t$. Hence by induction hypothesis

(1.21) $\{e\}_t^{(A \cup B_0)_t}(p) = \{e\}_t^{(A \cup B_1)_t}(p) = q$,

and one of the computations is A-correct. By symmetry we can assume w.l.o.g. that $\{e\}_t^{(A \cup B_0)}(p)$ is A-correct. The choice of t and s implies

(1.22) $\forall s'$ ($t < s' < s \implies r(e,s') \geq t$).

We now distinguish the following two cases.

Case 1 : $\{e\}_t^{(A \cup B_0)_t}(p)$ is destroyed at stage $t+1$. Since the computation is A-correct, a follower $x < u(e,(A \cup B_0)_t,p,t)$ of a requirement $P_{2e'}$ is enumerated in B_0 at stage $t+1$. Then all followers of lower priority requirements are cancelled at stage $t+1$, existing followers of higher priority requirements are less than x, and existing followers of $P_{2e'}$ different from x are never enumerated in a set. I.e. after stage $t+1$ only followers y s.t. $y < x$ or $y \geq \langle 0,t+1 \rangle \geq t+1$ can be enumerated in A, B_0, or B_1. By the A-correctness of $\{e\}_t^{(A \cup B_0)_t}(p)$ and $x < u(e,(A \cup B_0)_t,p,t)$, it follows that no follower, and therefore no number, less than $t+1$ is enumerated into A after stage t. In particular the computation $\{e\}_t^{(A \cup B_1)_t}(p)$ is A-correct. Note that at stage $t+1$ no number is enumerated in B_1. With (1.22)

follows that no number less than $u(e,(A\cup B_1)_t,p,t) \leq t$ is enumerated into B_1 at a stage s' s.t. $t+1 \leq s' \leq s$. Hence

$$\{e\}_s^{(A\cup B_1)_s}(p) = \{e\}_t^{(A\cup B_1)_t}(p)$$

via the same computation and therefore (1.20) holds for $i = 1$.

Case 2 : $\{e\}_t^{(A\cup B_0)_t}(p)$ is not destroyed at stage $t+1$. Then, by (1.22)

$$\{e\}_s^{(A\cup B_0)_s}(p) = \{e\}_t^{(A\cup B_1)_t}(p)$$

via the same computation and thus (1.20) holds for $i = 0$.

As seen before, lemma 1.3 and 1.6 imply that $\underset{\sim}{a}$ is branching. That $\underset{\sim}{a} \in NC$ follows from lemmas 1.4 and 1.5. This completes the proof of theorem 1.1.

Remarks. In the above proof the usual requirements to make $\underset{\sim}{a}$ low and requirements to avoid the cone of a given nonzero r.e. degree $\underset{\sim}{c}$ can be added. This yields

$\forall \underset{\sim}{c} > \underset{\sim}{0}\ \exists \underset{\sim}{a}$ ($\underset{\sim}{c} \not\leq \underset{\sim}{a}$, $\underset{\sim}{a}$ low, $\underset{\sim}{a} \in NC$, and $\underset{\sim}{a}$ branching).

Further, similar to Thomason [32], the branching degree construction can be extended to prove the following embedding result :

Let $\underset{\sim}{c} > \underset{\sim}{0}$ and let L be any countable distributive lattice. Then there exists an embedding of L in the low noncappable degrees which preserves suprema and infima.

In § 6 we show that every such lattice is embeddable in the noncappable degrees by a map which preserves suprema, infima, and greatest element.

§ 2 EFFECTIVELY NONCAPPABLE DEGREES.

In this section we introduce a subclass of the noncappable degrees, the effectively noncappable degrees. We show that this class has some nice properties from which it is unknown if the noncappable degrees have them. It is also unknown if both classes coincide, though all known constructions of noncappable degrees yield effectively noncappable degrees.

As NC the set ENC of the effectively noncappable degrees is a filter in $\underset{\sim}{R}$. In addition

$$\forall \underset{\sim}{a},\underset{\sim}{b} \in \mathrm{ENC}\ \exists\, \underset{\sim}{c} \in \mathrm{ENC}\ (\ \underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b}\ )$$

(Proposition 2.2). If NC has this property too, is unknown. Another open question concerning noncappable degrees, namely if strictly below any noncappable degree an other noncappable degree exists, is answered positively for effectively noncappable degrees (Corollary 2.2). Indeed we prove Sacks- and Robinson-splitting for effectively noncappable degrees (Theorems 2.1 and 2.2). Consequences are that ENC (NC) is no lattice (Corollary 2.3) and that ENC (NC) is an automorphism basis in $\underset{\sim}{R}$ (Corollary 2.6).

Definition 2.1. (a) An r.e. degree $\underset{\sim}{a}$ is effectively noncappable if there exists a sequence of uniformly r.e. sets A_e, $e < \omega$, s.t. for any set $A \in \underset{\sim}{a}$:

(2.1) $\forall e\ (\ A_e \leq_T A$ uniformly in e),

(2.2) $\forall e\ (\ A_e \leq_T W_e$ uniformly in e), and

(2.3) $\forall e\ (\ W_e$ nonrecursive $\Longrightarrow\ A_e$ nonrecursive).

(b) An r.e. set A is effectively noncappable if

(2.4) $\forall e\ (\ A^{(e)} \leq_T W_e$ uniformly in e) and

(2.5) $\forall e\ (\ W_e$ nonrecursive $\Longrightarrow\ A^{(e)}$ nonrecursive).

Let $\underline{ENC}$ be the set of the effectively noncappable degrees and let $\underline{ENC_0}$ be the set $ENC \cup \{\underset{\sim}{0}\}$.

Proposition 2.1. An r.e. degree is effectively noncappable iff it contains an effectively noncappable set.

Proof. Let $\underset{\sim}{a} \in ENC$. Choose a sequence of uniformly r.e. sets A_e, $e < \omega$, which satisfy (2.1) - (2.3), and fix an index e' s.t. $W_{e'} \in \underset{\sim}{a}$. Define a set A by

$$A^{(e)} = \begin{cases} \{\langle e,x\rangle : x \in A_e\} & \text{if } e \neq e' \\ \{\langle e,x\rangle : x \in W_{e'}\} & \text{if } e = e'. \end{cases}$$

Then $dg(A) = \underset{\sim}{a}$ and A is effectively noncappable.
Now fix an effectively noncappable set A and let $\underset{\sim}{a} = dg(A)$. Set $A_e = A^{(e)}$ for all $e < \omega$. Then $\{A_e : e < \omega\}$ is a sequence of uniformly r.e. sets which satisfy (2.1) - (2.3). Hence $\underset{\sim}{a}$ is effectively noncappable.

Proposition 2.2.

(a) $ENC \subseteq NC$

(b) $\underset{\sim}{0}' \in ENC$

(c) $\forall \underset{\sim}{a},\underset{\sim}{b} \in ENC \; \exists \underset{\sim}{c} \in ENC \; (\underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b})$

(d) ENC is a filter in $\underset{\sim}{R}$. I.e. for all $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

(d1) $(\underset{\sim}{a} \in ENC \text{ and } \underset{\sim}{a} < \underset{\sim}{b}) \Longrightarrow \underset{\sim}{b} \in ENC$ and

(d2) $(\underset{\sim}{a},\underset{\sim}{b} \in ENC \text{ and } \underset{\sim}{a} \cap \underset{\sim}{b} \text{ exists}) \Longrightarrow$

$$\Longrightarrow \underset{\sim}{a} \cap \underset{\sim}{b} \in ENC.$$

Proof. (a): By definitions.

(b): $K = \{\langle e,x\rangle : x \in W_e\}$ is a complete effectively noncappable set.

(c) : Fix $\underset{\sim}{a},\underset{\sim}{b} \in ENC$ and choose effectively noncappable sets $A \in \underset{\sim}{a}$ and $B \in \underset{\sim}{b}$. Further fix a recursive function f such that $A^{(e)} = W_{f(e)}$ for all $e < \omega$. Define a set C by $C^{(e)} = \{\langle e,x\rangle : \langle f(e),x\rangle \in B\}$. It is easy to see that C is effectively noncappable and $C \leq_T A,B$. Hence, for $\underset{\sim}{c} = dg(C)$, $\underset{\sim}{c} \in ENC$ and $\underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b}$.

(d) : (d1) follows from the definition and the (uniform) transitivity of $\leq_T$ immediately. To see (d2), fix $\underset{\sim}{a},\underset{\sim}{b} \in ENC$ s.t. $\underset{\sim}{a} \cap \underset{\sim}{b}$ exists. By (c) exists $\underset{\sim}{c} \in ENC$ such that $\underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b}$, i.e. s.t. $\underset{\sim}{c} \leq \underset{\sim}{a} \cap \underset{\sim}{b}$. Hence, by (d1), $\underset{\sim}{a} \cap \underset{\sim}{b} \subseteq ENC$.

Notice that parts (c) and (d) of the proposition imply that two degrees $\underset{\sim}{a},\underset{\sim}{b} \in ENC_0$ have an infimum in $\underset{\sim}{R}$ iff they have one in ENC_0 and if it exists it is the same in both structures. Hence ENC_0 is a sub-semilattice of the r.e. degrees.

Theorem 2.1 (Sacks splitting for effectively noncappable sets). Let A be an effectively noncappable set and $C_0,\ldots,C_n$, $n < \omega$, be nonrecursive r.e. sets. Then there exist r.e. sets B_0 and B_1 s.t.

(i) $A = B_0 \cup B_1$ and $B_0 \cap B_1 = \emptyset$,

(ii) B_0 and B_1 are effectively noncappable,

(iii) $C_i \not\leq_T B_0,B_1$ for all $i \leq n$, and

(iv) B_0 and B_1 are low.

The following corollary is the degree-theoretic formulation of theorem 2.1.

Corollary 2.1. Let $\underset{\sim}{a} \in ENC$ and $\underset{\sim}{c}_0, \ldots, \underset{\sim}{c}_n > \underset{\sim}{0}$ ($n < \omega$). Then there exist r.e. degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ s.t.

(i) $\underset{\sim}{a} = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$,

(ii) $\underset{\sim}{b}_0, \underset{\sim}{b}_1 \in ENC$,

(iii) $\underset{\sim}{c}_i \notin \underset{\sim}{b}_0, \underset{\sim}{b}_1$ for all $i \leq n$, and

(iv) $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ are low.

Corollary 2.2.

(a) $\forall \underset{\sim}{c}\ (\underset{\sim}{c} \neq \underset{\sim}{0}, \underset{\sim}{0}' \implies \exists \underset{\sim}{b} \in ENC\ (\underset{\sim}{c} \mid \underset{\sim}{b}$ and $\underset{\sim}{b}$ is low)).

(b) $\forall \underset{\sim}{a} \in ENC\ \exists \underset{\sim}{b} \in ENC\ (\underset{\sim}{b} < \underset{\sim}{a}$ and $\underset{\sim}{b}$ is low).

Proof. (a): Fix $\underset{\sim}{c} \neq \underset{\sim}{0}, \underset{\sim}{0}'$. Since, by proposition 2.2 (b), $\underset{\sim}{0}' \in ENC$, we can apply corollary 2.1 to $\underset{\sim}{a} = \underset{\sim}{0}'$ and $\underset{\sim}{c}_0 = \underset{\sim}{c}$ ($n=0$). This yields low effectively noncappable degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ such that $\underset{\sim}{0}' = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ and $\underset{\sim}{c} \notin \underset{\sim}{b}_0, \underset{\sim}{b}_1$. Hence it suffices to show that $\underset{\sim}{b}_0 \mid \underset{\sim}{c}$ or $\underset{\sim}{b}_1 \mid \underset{\sim}{c}$. Assume that $\underset{\sim}{b}_0 \nmid \underset{\sim}{c}$. Then, since $\underset{\sim}{c} \notin \underset{\sim}{b}_0$, $\underset{\sim}{b}_0 < \underset{\sim}{c}$. Since $\underset{\sim}{c} \neq \underset{\sim}{0}'$ and $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1 = \underset{\sim}{0}'$, it follows that $\underset{\sim}{b}_1 \notin \underset{\sim}{c}$, and thus, by $\underset{\sim}{c} \notin \underset{\sim}{b}_1$, that $\underset{\sim}{b}_1 \mid \underset{\sim}{c}$.

(b): Let $\underset{\sim}{a} \in ENC$ be given. Apply corollary 2.1 (for $\underset{\sim}{c}_0 = \underset{\sim}{a}$) to get low effectively noncappable degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ s.t. $\underset{\sim}{a} = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ and $\underset{\sim}{a} \notin \underset{\sim}{b}_0, \underset{\sim}{b}_1$. Then, for $\underset{\sim}{b} = \underset{\sim}{b}_0$, $\underset{\sim}{b} \in ENC$, $\underset{\sim}{b} < \underset{\sim}{a}$, and $\underset{\sim}{b}$ is low.

Corollary 2.3. ENC (NC) is no lattice.

Proof. Apply corollary 2.1 to $\underset{\sim}{a} = \underset{\sim}{0}'$ to get two low (effectively) noncappable degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ such that $\underset{\sim}{0}' = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$. By Lachlan [4], $\underset{\sim}{b}_0 \cap \underset{\sim}{b}_1$ doesn't exist. Hence, by proposition 2.2 (d) (proposition 1.1), $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ have no infimum in ENC (NC).

Proof of theorem 2.1. The proof combines Sacks' method to split an r.e. set (cf. e.g. [26], p.516) with the construction of an effectively noncappable set. The latter is the variant of the Yates noncappable degree construction which we used already in the proof of theorem 1.1.

Let $n < \omega$ and sets A, C_0, ... , C_n as in the premise of the theorem be given. Choose a one-one recursive function f which enumerates A and an effective enumeration $\{C_{i,s} : s < \omega\}$ of C_i ($i \leq n$). We will construct r.e. sets B_0 and B_1 which satisfy the conditions (i) through (iv) in the theorem.

At stage s+1 of the construction, f(s) will be enumerated in either B_0 or B_1, and no other number will be enumerated in B_0 or B_1 at that stage. It follows that $A = B_0 \cup B_1$, $B_0 \cap B_1 = \emptyset$, and

(2.6) $\forall j \leq 1 \forall e$ ($B_j^{(e)} \leq_T A^{(e)}$ uniformly in e).

(To decide recursively in $A^{(e)}$ if $x \in B_j^{(e)}$, for given x, first recursively in $A^{(e)}$ decide if $x \in A^{(e)}$. Is this not the case, $x \notin B_j^{(e)}$. Otherwise, effectively find s such that $x = f(s)$. Then $x \in B_j^{(e)}$ iff $x \in B_{j,s+1}^{(e)}$.)

Since (2.4) holds for A (and since the relation $\leq_T$ is uniformly transitive), (2.6) implies

(2.7) $\forall j \leq 1 \forall e$ ($B_j^{(e)} \leq_T W_e$ uniformly in e).

Hence, to make B_0 and B_1 effectively noncappable, it suffices to show

(2.8) $\forall j \leq 1 \forall e$ (W_e nonrecursive $\Longrightarrow$ $B_j^{(e)}$ nonrecursive).

Since A satisfies (2.5), (2.8) follows from

(2.9) $\forall j \leq 1 \forall e$ ($A^{(e)}$ nonrecursive $\Longrightarrow$ $B_j^{(e)}$ nonrecursive).

To guarantee (2.9) we meet the requirements

S_{2e+j} : $A^{(e_0)}$ nonrecursive $\Longrightarrow$ $B_j^{(e_0)} \neq \{e_1\}$

for all $j \leq 1$ and $e, e_0, e_1 < \omega$ s.t. $e = \langle e_0, e_1 \rangle$.

To satisfy condition (iii) of the theorem, we use the Sacks preservation strategy. We have to meet the negative requirements

$$N_{\langle e,i,j\rangle} : \quad C_i \neq \{e\}^{B_j}$$

for all $e < \omega$, $i \leq n$ and $j \leq 1$. To $N_{\langle e,i,j\rangle}$ a (recursive) lenght-function l and (recursive) restraint-function r are defined by

$$l(\langle e,i,j\rangle,s) = \max\{x : \forall y < x \ (C_{i,s}(y) = \{e\}_s^{B_j,s}(y))\}$$

and

$$r(\langle e,i,j\rangle,s) = \max\{u(e,B_{j,s},x,s) : x \leq l(\langle e,i,j\rangle,s)\}.$$

Let $Z = \{x : \exists e < \omega, i \leq n, j \leq 1 \ (x = \langle e,i,j\rangle)\}$, i.e. the set of indices of the negative requirements.

The way the requirements $N_{\langle e,i,j\rangle}$ are met will automatically insure that the sets B_0 and B_1 are low.

The requirement S_{2e+j} <u>requires attention</u> at stage s+1, if

(2.10) $\forall x \ (\{e_1\}_s(x) = 0 \implies B_{j,s}^{(e_0)}(x) = 0)$,

(2.11) $f(s) \in \omega^{(e_0)}$,

(2.12) $f(s) \geqslant \max\{r(x,s) : x \leq 2e+j \text{ and } x \in Z\}$, and

(2.13) $\{e_1\}_s(f(s)) = 0$.

We can now describe the <u>construction</u>:

<u>Stage s=0</u>. Do nothing.

<u>Stage s+1</u>. If a requirement S_{2e+j} requires attention at stage s+1, let $2e_0+j_0$ ($j_0 \leq 1$) be the least index of such a requirement and put $f(s)$ into B_{j_0} (We say : $S_{2e_0+j_0}$ is <u>active</u> at stage s+1.).

Otherwise fix the least $\langle e,i,j\rangle \notin Z$ s.t. $f(s) < r(\langle e,i,j\rangle,s)$, and put $f(s)$ into B_{1-j}. If no such $\langle e,i,j\rangle$ exists, put $f(s)$ into B_0.

Notice that the construction is effective, i.e. that B_0 and B_1 are r.e., and that $B_0 \cup B_1 = A$, $B_0 \cap B_1 = \emptyset$, and that (2.6) is satisfied. Hence the correctness of the construction will follow from the subsequent lemmata.

Lemma 2.1. For all $e < \omega$, $j \leq 1$, S_{2e+j} requires attention only finitely often.

Proof. By construction and condition (2.10), a requirement which is active at a stage s doesn't require attention at any stage $> s$. By induction on 2e+j, this easily implies lemma 2.1.

Lemma 2.2. For all $x \in Z$:

(a) If $x = \langle e,i,j \rangle$, then the set

$$I_x = \{ s : \exists y (y \in B_{j,s+1} \setminus B_{j,s} \text{ and } y < r(x,s)) \}$$

is finite.

(b) N_x is met.

(c) $\lim_s r(x,s) < \omega$ exists.

Let $r(x) = \lim_s r(x,s)$ for all $x \in Z$.

Lemma 2.2 is proved by the standard arguments. We follow Soare [26], pp 514-16.

Proof by simulataneous induction on x. For the inductive step fix $x = \langle e,i,j \rangle \in Z$.

(a) By induction hypothesis and lemma 2.1 there is a stage s_0 such that

$\forall s \geq s_0 \forall y < x (y \in Z \Rightarrow r(y,s) = r(y))$,

$\forall s \geq s_0 (f(s) \geq \max \{ r(y) : y < x \text{ and } y \in Z \})$, and

no requirement S_y, $y < x$, requires attention after stage s_0.

Then $\forall y (y \in I_x \Rightarrow y < s_0)$.

(b) Assume for a contradiction that $C_i = \{e\}^{B_j}$. Then

$\lim_s 1(x,s) = \omega$. Fix s_0 as in the proof of (a). We show that C_i is recursive contrary to hypothesis. To compute $C_i(p)$, for given $p < \omega$, find $s_1 \geq s_0$ such that $1(x,s_1) > p$. We claim that

(2.14) $\forall y \leq p \; (\; C_{i,s_1}(y) = C_i(y) \;)$,

and hence that C_i is recursive.

Assume for a contradiction that (2.14) fails, and let y_0 be the least number y s.t. $C_{i,s_1}(y) \neq C_i(y)$. Since

$\forall y < y_0 \forall s \geq s_1 \; (\; C_{i,s}(y) = C_i(y) \;)$,

it is seen, by induction on $s \geq s_1$, that

$\forall s \geq s_1 \; (\; 1(x,s) > y_0$ and $r(x,s) \geq u(e,B_{j,s},y_0,s) =$

$= u(e,B_{j,s_1},y_0,s_1) \;)$,

and thus, by choice of $s_1 \geq s_0$, that the computation $\{e\}^{B_{j,s_1}}(y_0)$ is B_j-correct. Hence $C_i(y_0) \neq C_{i,s_1}(y_0) = \{e\}^{B_{j,s_1}}(y_0) =$ $= \{e\}^{B_j}(y_0)$ contrary to the assumption that $C_i = \{e\}^{B_j}$.

(c) By (b), fix p minimal such that $C_i(p) \neq \{e\}^{B_j}$, and choose s_0 such that $s_0 > \max(I_x)$,

$\forall s \geq s_0 \forall y < p \; (\; \{e\}^{B_{j,s}}(y) = \{e\}^{B_j}(y) \;)$, and

$\forall s \geq s_0 \forall y \leq p \; (\; C_{i,s}(y) = C_i(y) \;)$.

Case 1 : $\forall s \geq s_0 \; (\; \{e\}^{B_{j,s}}(p) \uparrow \;)$. Then $r(x) = r(x,s_0)$.

Case 2 : $\exists s \geq s_0 \; (\; \{e\}^{B_{j,s}}(p) \downarrow \;)$. Then fix such an s. By induction on $t \geq s$, the computation $\{e\}^{B_{j,s}}(p)$ is B_j-correct. Hence

$\forall t \geq s \; (\; 1(x,t) = p$ and $r(x,t) = r(x,s) \;)$,

i.e. $r(x) = r(x,s)$.

Lemma 2.3. The sets B_0 and B_1 are low.

Proof. Fix a recursive function g s.t. (for all $X \subseteq \omega$ and $x < \omega$)

$$\{g(e)\}^X(x) = \begin{cases} C_0(0) & \text{if } x = 0 \text{ and } \{e\}^X(e) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Then

(2.15) $\forall j \leq 1 \forall e$ ($B'_j(e) = 0 \iff \lim_s 1(\langle g(e),0,j\rangle,s) = 0$)

and

(2.16) $\forall j \leq 1 \forall e$ ($B'_j(e) = 1 \iff \lim_s 1(\langle g(e),0,j\rangle,s) = 1$).

(Proof of (2.15): Assume that $B'_j(e) = 0$, i.e. that $\{e\}^{B_j}(e)\uparrow$. Let $s_o = \max(I_{\langle g(e),0,j\rangle})+1$. Then if $\{g(e)\}^{B_j}_{t}(e)\downarrow$ for any $t \geq s_o$, this computation is B_j-correct by construction, i.e., by definition of g, $\{e\}^{B_j}(e)\downarrow$ contrary to assumption. Hence $\forall t \geq s_o$ ($1(\langle g(e),0,j\rangle,s) = 0$).

Now assume that $\lim_s 1(\langle g(e),0,j\rangle,s) = 0$. Then, by definition of g, $\{g(e)\}^{B_j}(0)\uparrow$ and $\{e\}^{B_j}(e)\uparrow$, i.e. $B'_j(e) = 0$.

(2.16) is proved similarly.)

By (2.15) and (2.16), B'_j is the limit of a recursive function. It follows that B'_j is a Δ^0_2-set, and thus that $B'_j \leq_T \emptyset'$. Hence B_j is low ($j = 0,1$).

Lemma 2.4. For all $e < \omega$ and $j \leq 1$, S_{2e+j} is met.

Proof. Fix $2e+j$, $j \leq 1$, and e_o, e_1 s.t. $e = \langle e_o, e_1\rangle$. Assume that $B_j^{(e_o)} = \{e_1\}$. We have to show that $A^{(e_o)}$ is recursive.

Let $D = A^{(e_o)} \setminus B^{(e_o)}$. Then, by $B_j^{(e_o)} = \{e_1\}$,

$\forall x$ ($x \in D \iff$ ($x \in \omega^{(e_o)}$, $\exists s$ ($f(s) = x$), and $\{e_1\}(x) = 0$)),

and (2.10) holds for all stages s.

By lemmata 2.1 and 2.2, fix a stage s_o such that

$\forall s \geq s_o \forall x \leq 2e+j$ ($x \in Z \implies r(x,s) = r(x)$),

$\forall s \geq s_o$ ($f(s) \geq \max\{r(x) : x \leq 2e+j$ and $x \in Z\}$), and

S_{2e+j} doesn't require attention after stage s_o.

Hence there is no $s \geq s_o$ s.t. $f(s) \in \omega^{(e_o)}$ and $\{e_1\}_s(f(s)) = 0$ (otherwise S_{2e+j} would require attention at that stage s+1). It follows

(2.17) $\forall x$ ($x \in D \iff$ ($x \in \omega^{(e_o)}$, $\exists s \leq \max\{s_o, \mu t(\{e_1\}_t(x)\downarrow)\}$(

($f(s) = x$), and $\{e_1\}(x) = 0$).

Since $\{e_1\}$ is total, (2.17) implies that D is recursive. It follows that $A^{(e_o)}$, as union of the two recursive sets D and $B_j^{(e_o)}$, is recursive too.

This completes the proof of theorem 2.1.

Theorem 2.2 (Robinson splitting for effectively noncappable sets). Let $n < \omega$ and r.e. sets A, C_o, ..., C_n and D be given such that A is effectively noncappable, $D \leq_T A$, D is low, and $C_i \not\leq_T D$ for all $i \leq n$. Then there exist r.e. sets B_o and B_1 such that

(i) $A = B_o \cup B_1$ and $B_o \cap B_1 = \emptyset$,

(ii) B_o and B_1 are effectively noncappable,

(iii) $C_i \not\leq_T B_o \oplus D, B_1 \oplus D$ for all $i \leq n$, and

(iv) $B_o \oplus D$ and $B_1 \oplus D$ are low.

The degree-theoretic formulation of theorem 2.2 is given in the next corollary.

Corollary 2.4. Let $n < \omega$ and r.e. degrees $\underset{\sim}{a}$, $\underset{\sim}{c}_o$, ..., $\underset{\sim}{c}_n$ and $\underset{\sim}{d}$ be given such that $\underset{\sim}{a} \in ENC$, $\underset{\sim}{d} < \underset{\sim}{a}$, $\underset{\sim}{d}$ is low, and $\underset{\sim}{c}_i \not\leq \underset{\sim}{d}$ for all $i \leq n$. Then there exist r.e. degrees $\underset{\sim}{b}_o$ and $\underset{\sim}{b}_1$ such that

(i) $\underset{\sim}{a} = \underset{\sim}{b}_o \cup \underset{\sim}{b}_1$,

(ii) $\underset{\sim}{b}_o, \underset{\sim}{b}_1 \in ENC$,

(iii) $\underset{\sim}{d} \leq \underset{\sim}{b}_o, \underset{\sim}{b}_1$,

(iv) $\underset{\sim}{c}_i \not\leq \underset{\sim}{b}_o, \underset{\sim}{b}_1$, and

(v) $\underset{\sim}{b}_o$ and $\underset{\sim}{b}_1$ are low.

Corollary 2.5. If $\underset{\sim}{a} \in ENC$, $\underset{\sim}{b} < \underset{\sim}{a}$, $\underset{\sim}{b}$ is low and $\underset{\sim}{c} \not\leq \underset{\sim}{b}$, then there

exists $\underset{\sim}{d} \in ENC$ such that $\underset{\sim}{b} < \underset{\sim}{d} < \underset{\sim}{a}$ and $\underset{\sim}{c} \not\leq \underset{\sim}{d}$. If in addition $\underset{\sim}{a} \leq \underset{\sim}{c}$, then $\underset{\sim}{c} \mid \underset{\sim}{d}$.

Corollary 2.5 follows from corollary 2.4 similarly to the way corollary 2.2 follows from corollary 2.1.

Corollary 2.6. ENC (NC) is an automorphism basis for $\underset{\sim}{R}$.

Proof. Since ENC $\subseteq$ NC, it suffices to show that ENC is an automorphism basis.
Fix an automorphism f and a degree $\underset{\sim}{a}$ such that $f(\underset{\sim}{a}) \neq \underset{\sim}{a}$. We have to show that, for some $\underset{\sim}{b} \in ENC$, $f(\underset{\sim}{b}) \neq \underset{\sim}{b}$.
Since the low degrees are an automorphism basis, we may assume that $\underset{\sim}{a}$ is low.
Case 1 : $f(\underset{\sim}{a}) \not\leq \underset{\sim}{a}$. By corollary 2.5 there exists a degree $\underset{\sim}{b} \in ENC$ such that $\underset{\sim}{a} \leq \underset{\sim}{b}$ and $f(\underset{\sim}{a}) \not\leq \underset{\sim}{b}$. Hence, since f is an automorphism, $f(\underset{\sim}{a}) \leq f(\underset{\sim}{b})$ and thus $f(\underset{\sim}{b}) \neq \underset{\sim}{b}$.
Case 2 : $f(\underset{\sim}{a}) \leq \underset{\sim}{a}$. Then, by assumption, $f(\underset{\sim}{a}) < \underset{\sim}{a}$. Since f^{-1} is an automorphism, $\underset{\sim}{a} = f^{-1}(f(\underset{\sim}{a})) < f^{-1}(\underset{\sim}{a})$. Hence we can apply case 1 (to f^{-1}) to get a degree $\underset{\sim}{b} \in ENC$ such that $f^{-1}(\underset{\sim}{b}) \neq \underset{\sim}{b}$. It follows that $\underset{\sim}{b} = f(f^{-1}(\underset{\sim}{b})) \neq f(\underset{\sim}{b})$.

Corollary 2.7. Let $\underset{\sim}{a} \in ENC$, $\underset{\sim}{b} < \underset{\sim}{a}$, and $\underset{\sim}{b}$ be low. Then every countable upper-semilattice is embeddable into the effectively noncapable degrees of the interval $[\underset{\sim}{a},\underset{\sim}{b}] = \{\underset{\sim}{c} : \underset{\sim}{a} \leq \underset{\sim}{c} \leq \underset{\sim}{b}\}$ by a map which preserves suprema and greatest element.

Proof. By corollary 2.5, find $\underset{\sim}{c} \in ENC$ such that $\underset{\sim}{b} \leq \underset{\sim}{c} < \underset{\sim}{a}$ and apply Robinson [14], Corollary 11 to $[\underset{\sim}{c},\underset{\sim}{a}]$.

Proof of theorem 2.2. A technique of Robinson [14] to handle low degrees is applied to the proof of theorem 2.1.

Fix $n < \omega$ and r.e. sets A, C_0, ... , C_n and D as in the premise of the theorem. Choose the function f and $\{C_{i,s} : s < \omega\}$ as in the proof of theorem 2.1, and let $\{D_s : s < \omega\}$ be an effective enumeration of D.

A is split into two r.e. sets B_0 and B_1 as in the preceding proof. Hence $A = B_0 \cup B_1$, $B_0 \cap B_1 = \emptyset$, and (2.6) will follow as there. To make B_0 and B_1 effectively noncappable, we will meet the requirements S_{2e+j} of the proof of theorem 2.1 for all $e < \omega$ and $j \leq 1$. S_{2e+j} requires attention as there.

The negative requirements, which will insure (iii), are

$$N_{\langle e,i,j\rangle} : \quad C_i \neq \{e\}^{B_j} \oplus D$$

for all $e < \omega$, $i \leq n$, $j \leq 1$. Z is defined as in the previous proof. A recursive length-function l is defined by

$$l(\langle e,i,j\rangle,s) = \max\{x : \forall y < x \ (\ C_{i,s}(y) = \{e\}^{B_j}_s,s \oplus D_s \ )\} .$$

A modified length-function m and a restraint function r are defined during the construction.

An obstacle to the strategy to meet requirement $N_{\langle e,i,j\rangle}$ by preserving agreement of C_i and $\{e\}^{B_j} \oplus D$ on initial segments is the lack of control over D. It may happen now that $\{e\}^{B_j} \oplus D(0)\uparrow$ while, for infinitely many $s < \omega$, $\{e\}^{B_j}_s,s \oplus D_s(0)\downarrow$ and $\lim_s u(e,B_{j,s} \oplus D_s,0,s) = \omega$. Hence if we put restraints on B_j as before, the restraints by the single requirement $N_{\langle e,i,j\rangle}$ may be unbounded, and we may not be able to satisfy the positive requirements S_{2e+j}. If we protect only D-correct computations, however, for a single requirement bounded restraints suffice as before.

The problem is of course that in general it is undecidable if a computation is D-correct. The lowness of D, however, will allow us to guess sufficiently successfully the D-correctness of computations. For this purpose we use the following fact (cf. Soare [27], Proposition 2.5 and theorem 2.7): Since D is low there exists a recursive function h such that

(2.18) $\forall e \; (\; W_e \cap \{ x : \mathbb{D}_x \subseteq \overline{D} \} \subseteq W_{h(e)} \;)$ and

(2.19) $\forall e \; (\; W_e \cap \{ x : \mathbb{D}_x \subseteq \overline{D} \}$ finite $\Longrightarrow$ $W_{h(e)}$ finite).

To apply (2.18) and (2.19) we will use a "test set" V, which is enumerated during the construction. It will be necessary to know an r.e. index of V in advance. Hence we do ω many constructions and assume in the n-th construction that (for the set V enumerated in this construction) $V = W_n$. By the recursion theorem, for some n this assumption will be correct. We will show that the sets B_0 and B_1 enumerated in this construction have the desired properties.

Let $\lambda x,y([x]_y)$ be a recursive function s.t., for all x and y, $W_{[x]_y} = \{ z : z \in W_x^{(y)} \}$.
We can now describe the construction.

n-construction ($n < \omega$):

Stage s=0. For all $x \in Z$, set $m(x,0) = r(x,0) = 0$.

Stage s+1. The stage consists of two steps. Numbers are put into B_0 and B_1 only during step 1. Hence $B_{j,s+1}$ ($j = 0,1$) is the set of numbers enumerated in B_j by the end of step 1 of stage s+1. In the second step the functions m and r are defined.

Step 1. Do the same as at stage s+1 of the construction in the proof of theorem 2.1.

Step 2. For $x > s$ s.t. $x \in Z$, set $m(x,s+1) = r(x,s+1) = 0$. For $x \leq s$ s.t. $x \in Z$ and $x = \langle e,i,j \rangle$ for some $e,i,j < \omega$, and for all $y \leq l(x,s+1)$, let $u(y)$ be the canonical index of $\{ z : 2z+1 < \max \{ u(e,B_{j,s+1} \oplus D_{s+1},v,s+1) : v \leq y \}$ and $z \notin D_{s+1} \}$. Put $\langle x,u(y) \rangle$ into V, and simulataneously enumerate $W_{h([n]_x)}$ and D until either $u(y)$ shows up in $W_{h([n]_x)}$ or an element of $\mathbb{D}_{u(y)}$ shows up in D. (If none of both happens, stage s+1 is never completed.). If y_o is the greatest y such that $u(y)$ shows up in $W_{h([n]_x)}$, then set $m(x,s+1) = y_o+1$ and $r(x,s+1) = \max \{ u(e,B_{j,s+1} \oplus D_{s+1},v,s+1) : v \leq y_o \}$. If no such y_o exists, set $m(x,s+1) = r(x,s+1) = 0$.

The n-construction is effective uniformly in n, $n < \omega$. I.e. the sequence $\{ V^n : n < \omega \}$, where V^n is the set V constructed in the n-construction, is uniformly r.e.. Hence there exists a recursive function k s.t. $V^n = W_{k(n)}$. By the recursion theorem, for some n, $V^n = W_n$. For the remainder of the proof, fix such an n. We will show that the sets B_o and B_1 constructed in the n-construction have the desired properties.

First notice that every stage of the construction is completed: If in step 2 of a stage s+1 a number $\langle x,u(y) \rangle$ is put into $V = W_n$, then $u(y) \in W_{[n]_x}$. Hence, by (2.18), either $u(y) \in W_{h([n]_x)}$ or an element of $\mathbb{D}_{u(y)}$ is in D.

That the sets B_o and B_1 are r.e., $A = B_o \cup B_1$, $B_o \cap B_1 = \emptyset$, and that (2.6) holds, follows from the construction immediately. Hence, as in the proof of theorem 2.1, it suffices to prove the subsequent lemmata.

Lemma 2.5. For all $e < \omega$ and $j \leq 1$, S_{2e+j} requires attention at most finitely often.

Lemma 2.5 is proved as lemma 2.1.

Lemma 2.6. For all $x \in Z$:

(a) If $x = \langle e,i,j \rangle$, then the set

$$I_x = \{s : \exists y \, (\, y \in B_{j,s+1} \oplus \emptyset \setminus B_{j,s} \oplus \emptyset \text{ and } y < r(x,s) \,)\}$$

is finite.

(b) N_x is met.

(c) $\lim_s r(x,s) < \omega$ exists.

For all $x \in Z$ let $r(x) = \lim_s r(x,s)$.

Proof of lemma 2.6 by simultaneous induction on $x \in Z$. For the inductive step fix $x \in Z$, $x = \langle e,i,j \rangle$.

(a) This is proved as lemma 2.2(a).

(b) Assume for a contradiction that $C_i = \{e\}^{B_j \oplus D}$. Then $\lim_s l(x,s) = \omega$. By (a), fix $s_0 > \max(I_x)$. We show that $C_i \leq_T D$ contrary to hypothesis. To D-recursively compute $C_i(p)$, for given $p < \omega$, find $s_1 \geq s_0$ such that $l(x,s_1) > p$ and

$$\forall y \leq p \, (\, \{e\}^{B_j,s_1}_{s_1} \oplus D_{s_1} (y) \text{ is } \emptyset \oplus \text{D-correct}.$$

As in the proof of lemma 2.2(b) it is seen that

$$\forall y \leq p \, (\, C_{i,s_1} (y) = C_i (y) \,).$$

Hence $C_i \leq_T D$. Contradiction.

(c) By (b), fix p minimal s.t. $C_i(p) \neq \{e\}^{B_j \oplus D}(p)$, and, by (a), choose a stage s_0 s.t. $s_0 > \max(I_x)$ and s.t., for all $s \geq s_0$,

$$\forall y < p \, (\, \{e\}^{B_j,s}_s \oplus D_s \, (y) = \{e\}^{B_j \oplus D}(y) \text{ and}$$
$$\{e\}^{B_j,s}_s \oplus D_s \, (y) \text{ is } \emptyset \oplus \text{D-correct}), \text{ and}$$

$$\forall y \leq p \, (\, C_{i,s}(y) = C_i(y) \,).$$

Case 1 : There exists $s \geq s_0$ s.t. $\{e\}^{B_j,s}_s \oplus D_s \, (p) \downarrow$ and the computation is $\emptyset \oplus$D-correct. Then let s_1 be the least such stage. By induction on $s \geq s_1$,

$\forall s \geq s_1$ ($l(x,s) \geq p$, $m(x,s) \geq p+1$, $\{e\}_s^{B_j,s} \oplus D_s(p) =$
$= \{e\}_{s_1}^{B_j,s_1} \oplus D_{s_1}(p)$, and $r(x,s) \geq u(e,B_{j,s} \oplus D_s,p,s) =$
$= u(e,B_{j,s_1} \oplus D_{s_1},p,s_1)$).

Hence the computaion $\{e\}_{s_1}^{B_j,s_1} \oplus D_{s_1}(p)$ is $(B_j \oplus D)$-correct, i.e.

$\forall s \geq s_1$ ($C_{i,s}(p) \neq \{e\}_s^{B_j,s} \oplus D_s(p)$).

It follows that

$\forall s \geq s_0$ ($l(x,s) = p$, $m(x,s) = p+1$, and $r(x,s) = r(x,s_1)$).

Hence $r(x) = r(x,s_1)$.

Case 2 : Otherwise. If $\{e\}_s^{B_j,s} \oplus D_s(p) \downarrow$ for only finitely many s, fix the least $s_1 \geq s_0$ s.t. $\forall s \geq s_1$ ($\{e\}_s^{B_j,s} \oplus D_s(p) \uparrow$). Then $r(x) = r(x,s_1)$. Hence we can assume that there are infinitely many stages $s \geq s_0$ where $\{e\}_s^{B_j,s} \oplus D_s(p)$ is defined, and none of these computations is $\emptyset \oplus D$-correct.

Hence, by choice of s_0,

(2.20) $\forall y$ ($\langle x,y \rangle \in V \setminus V_{s_0} \implies \mathbb{D}_y \cap D \neq \emptyset$).

We claim that

(2.21) $r(x) = u$, where $u = \max \{ u(e,B_{j,s_0} \oplus D_{s_0},y,s_0) : y < p \}$.

Assume for a contradiction that (2.21) fails, i.e. that

$\forall s \exists t \geq s$ ($r(x,t) \neq u$).

Since, by definition of s_0, $r(x,s) \geq u$ for all $s \geq s_0$, it follows that there are infinitely many stages t such that $m(x,t) \geq p+1$. Hence, if $u(p,t)$ is the canonical index of the set

$\{ z : \exists y \leq p$ ($2z+1 < u(e,B_{j,t} \oplus D_t,y,t)$) and $z \notin D_t \}$,

there are infinitely many t such that

$\langle x,u(p,t) \rangle \in V$, $u(p,t) \in W_{h([n]_x)}$, and $\mathbb{D}_{u(p,t)} \cap D \neq \emptyset$.

Since $\mathbb{D}_{u(p,t)} \cap D_t = \emptyset$, this implies that the set $W_{h([n]_x)}$ is infinite. Since, by (2.20) and $V = W_n$, $W_{[n]_x} \cap D$ is finite, this contradicts (2.19).

Lemma 2.7. $B_0 \oplus D$ and $B_1 \oplus D$ are low.

Proof. Define g as in the proof of lemma 2.3 and define $\hat{m}$ by

$$\hat{m}(x,s) = \begin{cases} m(x,s)-1 & \text{if } m(x,s) > 0 \text{ and } \langle e\rangle_s^{B_j}, s^{\oplus D}s \ (m(x,s)-1)\uparrow \\ m(x,s) & \text{otherwise} \end{cases}$$

(for all $x \in Z$ s.t. $x = \langle e,i,j \rangle$). Then, for $j = 0,1$ and all e, $(B_j \oplus D)'(e) = \lim_s \hat{m}(\langle g(e),0,j\rangle,s)$.
This is seen as in the proof of lemma 2.3.

Lemma 2.8. For all $e > \omega$ and $j \leq 1$, S_{2e+j} is met.

Lemma 2.8 is proved as lemma 2.4.

This completes the proof of theorem 2.2.

Remarks. 1. Theorems 2.1 and 2.2 and their corollaries hold for a weaker notion of effectively noncappable, where conditions (2.2) and (2.4) are omitted. The proofs remain the same. (It is unknown, however, if proposition 2.2(c) holds for this weaker notion.)

2. As already mentioned, it is unknown if the noncappable degrees have the properties

(2.22) $\forall \underset{\sim}{a}\ (\ \underset{\sim}{a} \in NC \implies \exists \underset{\sim}{b} < \underset{\sim}{a}\ (\ \underset{\sim}{b} \in NC\ )\ )$ and

(2.23) $\forall \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{a},\underset{\sim}{b} \in NC \implies \exists \underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{c} \in NC)\ )$,

which we proved for effectively noncappable degrees in Corollary 2.2 and proposition 2.2(c), respectively. Fejer [1] proved that another subclass of NC, namely the wtt-cuppable degrees, satisfy (2.22), i.e. that

(2.24) $\forall \underset{\sim}{a}\ (\ \underset{\sim}{a}$ wtt-cuppable $\implies \exists \underset{\sim}{b} < \underset{\sim}{a}\ (\ \underset{\sim}{b} \in NC\ )\ )$.

His proof actually shows that every wtt-cuppable degree is effectively noncappable. Hence, by proposition 2.2(c),

(2.25) $\forall \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{a},\underset{\sim}{b}$ wtt-cuppable $\implies \exists \underset{\sim}{c} \leq \underset{\sim}{a},\underset{\sim}{b}\ (\ \underset{\sim}{c} \in NC\ )\ )$.

3. The degree $\mathfrak{g}$ constructed in the proof of theorem 1.1 is effectively noncappable. Hence

(2.26) There is a branching effectively noncappable degree. It follows that Shoenfield's conjecture restricted to ENC also fails.

§ 3 EVERY CONE CONTAINS AN INCOMPLETE NONCAPPABLE DEGREE.

The existence of nonzero cappable degrees easily implies that below every nonzero r.e. degree a nonzero cappable degree exists (cf. (1.9)). The dual question, whether above every incomplete r.e. degree an incomplete noncappable degree exists, doesn't allow such an easy answer.

Combining the Yates noncappable degree construction with a technique of Robinson to handle low degrees, Fejer got a partial answer : Above every low r.e. degree there exists an incomplete noncappable degree. We give a general positive answer :

(3.1) $\forall \underset{\sim}{a} < \underset{\sim}{0}' \exists \underset{\sim}{b}$ ($\underset{\sim}{b} \in NC$ and $\underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{0}'$).

In this section we give a direct proof of (3.1). In § 4 we will get a different proof, which gives some additional information (cf. Corollary 4.2).

Theorem 3.1. If $\underset{\sim}{a}$ is incomplete then there exists a noncappable degree $\underset{\sim}{b}$ such that $\underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{0}'$.

Proof. Fix $\underset{\sim}{a} < \underset{\sim}{0}'$. If $\underset{\sim}{a}$ is noncappable, by the density theorem, choose $\underset{\sim}{b}$ such that $\underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{0}'$. By proposition 1.1 $\underset{\sim}{b}$ is noncappable. Hence we can assume in the following that $\underset{\sim}{a}$ is cappable. Choose $\underset{\sim}{c} > \underset{\sim}{0}$ such that $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$. We will construct a noncappable degree $\underset{\sim}{b}$ s.t. $\underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}'$. Then by proposition 1.1 $\underset{\sim}{a} \oplus \underset{\sim}{b}$ is noncappable and $\underset{\sim}{a} < \underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}'$.

Fix r.e. sets $A \in \underset{\sim}{a}$ and $C \in \underset{\sim}{c}$ and effective enumerations $\{A_s : s < \omega\}$ and $\{C_s : s < \omega\}$ of A and C, respectively. An r.e. set B is constructed such that

(3.2) B is effectively noncappable

and

(3.3) $A \oplus B$ is incomplete.

Then $\underset{\sim}{b} = dg(B)$ has the above properties.

(3.2) is attacked as the similar conditions in the previous proofs. We meet the requirements

$$S_e : W_{e_0} \text{ nonrecursive} \Longrightarrow B^{(e_0)} \neq \{e_1\}$$

for all $e, e_0, e_1 < \omega$, where $e = \langle e_0, e_1 \rangle$, and insure, by permitting,

$$B^{(e)} \leq_T W_e \text{ uniformly in } e$$

for all $e < \omega$.

To make $A \oplus B$ incomplete, we construct an r.e. set D such that $D \not\leq_T A \oplus B$, i.e.

(3.4) For all $e < \omega$, $D \neq \{e\}^{A \oplus B}$.

The basic strategy to meet requirements of type (3.4), where A and B are sets which one is constructing, is that of Friedberg [2] and Muchnik [13] :

Wait for a stage s and a number x (which is an element of some specified set) such that

(3.5) $\{e\}_s^{A_s \oplus B_s}(x) = 0$.

Put x into D (i.e. $D(x)=1$) and put restraints on A and B to protect (3.5).

Here, however, A is a given set and hence a restraint can be put on B only, which doesn't suffice to make the strategy work. We will win the necessary control over A by simultaneously attacking the fact that $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$. We construct r.e. sets E_e for which we show :

(3.6) $D = \{e\}^{A \oplus B} \Longrightarrow \emptyset \leq_T E_e \leq_T A, C$.

Since the conclusion of (3.6) contradicts $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$, the premise must be false, i.e. (3.4) is true.

To be more precise : Instead of (3.4) we will meet the requirements

R_e : $D \neq \{e\}^{A \oplus B}$ or $E_{e_o} \neq \{e_1\}$

for all $e<\omega$, $e=\langle e_o, e_1\rangle$, and satisfy the conditions

(3.7) For all $e<\omega$, $E_e \leq_T A$

and

(3.8) For all $e<\omega$, if $D = \{e\}^{A \oplus B}$ then $E_e \leq_T C$.

To the requirement R_e a restraint function r is defined during the construction.

R_e requires attention at stage s+1 if $r(e,s) = 0$ and there exist numbers x and y such that

(3.9) for some x', $y'<\omega$, $x = \langle e,x'\rangle$ and $y = \langle e,y'\rangle$,

(3.10) $\{e_o\}_s^{A_s \oplus B_s}(x) = 0$,

(3.11) $y > u(e_o, A_s \oplus B_s, x, s)$,

(3.12) $\{e_1\}_s(y) = 0$,and

(3.13) $\exists z<y$ ($z \in C_{s+1} \setminus C_s$).

If R_e requires attention at stage s+1 and x and y satisfy (3.9) - (3.13) then R_e requires attention through $\langle x,y\rangle$ at stage s+1.

S_e requires attention at stage s+1 if

(3.14) $\forall x$ ($\{e_1\}_s(\langle e_o, e_1, x\rangle) = 0 \Rightarrow \langle e_o, e_1, x\rangle \notin B_s$)

and there exists a number x such that

(3.15) $x = \langle e_o, e_1, x'\rangle$ for some $x'<\omega$,

(3.16) $\{e_1\}_s(x) = 0$,

(3.17) $\exists y<x$ ($y \in W_{e_o,s+1} \setminus W_{e_o,s}$), and

(3.18) $x \geq \max\{r(i,s) : i \leq e\}$.

If S_e requires attention at stage s+1 and x satisfies (3.15)-(3.18) then S_e requires attention through x at stage s+1.

The priority ordering of the requirements is :

$R_0 > S_0 > \cdots > R_{n+1} > S_{n+1} > \cdots$.

Construction :

Stage 0. For all $e < \omega$, $r(e,0) = 0$.

Stage s+1. The stage consists of two steps.

Step 1. If no requirement requires attention then set $r(e,s+1) = r(e,s)$ for all $e < \omega$.
Otherwise let R be the highest priority requirement which requires attention at stage s+1. (We say R is active at stage s+1.) Distinguish the following two cases :

Case 1 : $R = R_e$. Let $\langle x,y \rangle$ be the least number through which R_e requires attention. Put x into D, appoint y as attacker of R_e, and set (for all $f < \omega$)

$$r(f,s+1) = \begin{cases} u(e_0, A_s \oplus B_s, x, s) & \text{if } f = e \\ r(f,s) & \text{otherwise.} \end{cases}$$

Case 2 : $R = S_e$. Let x be the least number through which S_e requires attention. Put x into B, cancel all attackers of requirements R_f, $f > e$, and set

$$r(f,s+1) = \begin{cases} r(f,s) & \text{if } f \leq e \\ 0 & \text{otherwise.} \end{cases}$$

Step 2. For all $e < \omega$, $e = \langle e_0, e_1 \rangle$, if y is an (uncancelled) attacker of R_e (at the end of step 1) and if $\exists z < y \ (z \in A_{s+1} \setminus A_s)$ then put y into E_{e_0}.
(Note that numbers are enumerated in A only during step 1. Hence A_{s+1} is the finite set of numbers put into A by the end of step 1 in stage s+1.)

Note that the construction is effective. Hence B, D, and E_e, $e < \omega$, are recursively enumerable sets. That the sets have the desired properties will follow from the subsequent lemmata.

Lemma 3.1. Every requirement requires attention at most finitely often.

Proof by induction on the priority rank of the requirement. For the inductive step fix a requirement R. By induction hypothesis choose a stage s_0 after which no higher priority requirement requires attention.

Case 1 : $R = R_e$. If R_e requires attention at a stage $s > s_0$ then R_e is active at stage s and (by choice of s_0)

$$\forall s' > s \ (\ r(e,s') = r(e,s) > 0 \ ).$$

Hence R_e doesn't require attention after stage s.

Case 2 : $R = S_e$, $e = \langle e_0, e_1 \rangle$. If S_e requires attention at a stage $s+1 > s_0$ then S_e is active at stage s+1. If $x = \langle e_0, e_1, x' \rangle$ is the least number through which S_e requires attention at stage s+1 then $x \in B_{s+1}$ and $\{e_1\}_s(x) = 0$. It follows that for no $s' > s$ (3.14) holds and thus S_e doesn't require attention after stage s+1.

Lemma 3.2. For all $e < \omega$, $\lim_s r(e,s) < \omega$ exists.

Proof. This follows from lemma 3.1 immediately.

Set $r(e) = \lim_s r(e,s)$.

Lemma 3.3. For all $e < \omega$, S_e is met.

Proof. Fix $e = \langle e_0, e_1 \rangle$ and assume that $B^{(e_0)} = \{e_1\}$. We have to show that W_{e_0} is recursive.

By lemmata 3.1 and 3.2 fix s_0 such that

$$\forall s \geq s_0 \, \forall y \leq e \ (\ r(y,s) = r(y) \ )$$

and S_e doesn't require attention after stage s_0.

Let $x_0 = \max \{r(y) : y \leq e\}$. Since $B^{(e_0)} = \{e_1\}$, (3.14) holds for all $s < \omega$ and, for all $x < \omega$, $\{e_1\}(\langle e_0, e_1, x\rangle) = 0$. Hence the function f defined by

$$f(x) = \mu s \geq s_0 \ (\{e_1\}_s(\langle e_0, e_1, x\rangle) = 0$$

is total and recursive. Since S_e doesn't require attention after stage s_0, it follows that for all $x \geq x_0$

$$W_{e_0, f(x)}[\langle e_0, e_1, x\rangle] = W_{e_0}[\langle e_0, e_1, x\rangle],$$

and hence that W_{e_0} is recursive.

Lemma 3.4. For all $e < \omega$, $B^{(e)} \leq_T W_e$ uniformly in e.

Proof. By permitting.

Lemma 3.5. For all $e < \omega$, R_e is met.

Proof. Fix $e = \langle e_0, e_1\rangle$.

Case 1 : $r(e) > 0$. Let s+1 be the last stage at which R_e requires attention and let $\langle x, y\rangle$ be the least number through which R_e requires attention at s+1. By $r(e) > 0$, R_e is active at stage s+1 and thus

$x \in D_{s+1}$, $\{e_0\}_s^{A_s \oplus B_s}(x) = 0$, $y \geq u(e_0, A_s \oplus B_s, x, s)$, and $\{e_1\}_s(y) = 0$. Further, since $r(e,t) = r(e) > 0$ for all $t > s+1$, no number less than $u(e_0, A_s \oplus B_s, x, s) = r(e, s+1)$ is enumerated into B after stage s, and y is attacker of R_e at every stage $> s+1$.

Hence either $A_s[y] \neq A[y]$, in which case y is enumerated in E_{e_0} at some stage $> s+1$ insuring $E_{e_0} \neq \{e_1\}$, or $A_s[y] = A[y]$, in which case the computation $\{e_0\}_s^{A_s \oplus B_s}(x) = 0$ is $A \oplus B$ - correct, i.e. $\{e_0\}^{A \oplus B}(x) = 0$ implying $D \neq \{e_0\}^{A \oplus B}$. In either case R_e is met.

Case 2 : $r(e) = 0$. Assume for a contradiction that R_e is not met, i.e. $D = \{e_0\}^{A \oplus B}$ and $E_{e_0} = \{e_1\}$.

Fix s_0 such that for all $s \geq s_0$, $r(e,s) = 0$. By lemma 3.1 choose

$s_1 \geq s_0$ and y_0 s.t. R_e doesn't require attention after stage s_1 and $\forall y \geq y_0$ ($\langle e,y\rangle \notin E_{e_0}$). Since $E_{e_0} = \{e_1\}$,

(3.19) $\forall y \geq y_0$ ($\{e_1\}(\langle e,y\rangle) = 0$).

By lemma 3.1 and $D = \{e_0\}^{A \oplus B}$, choose $x = \langle e,x'\rangle$ such that $\{e_0\}^{A \oplus B}(x) = 0$. Let $u = u(e_0, A \oplus B, x)$, let s_2 be a stage $\geq s_1$ s.t.

$\forall s \geq s_2$ ($\{e_0\}_s^{A_s \oplus B_s}(x) = 0$ and $u(e_0, A_s \oplus B_s, x, s) = u$),

and let $y_1 = \max\{y_0, u\}$. Further define a function f by

$$f(y) = \begin{cases} 0 & \text{if } y < y_1 \\ \mu s \geq s_2 \; (\{e_1\}_s(\langle e,y\rangle) = 0) & \text{otherwise.} \end{cases}$$

By (3.19), f is a total recursive function, and

(3.20) $\forall y \geq y_1$ ($C_{f(y)}[\langle e,y\rangle] = C[\langle e,y\rangle]$).

(Otherwise fix $y \geq y_1$ and $s \geq f(y)$ such that $\exists z < \langle e,y\rangle$ ($z \in C_{s+1} - C_s$). Then R_e requires attention through $\langle x, \langle e,y\rangle\rangle$ at stage $s+1 > s_0$ contrary to the choice of s_0.)

(3.20) implies that C is recursive contradicting $\underset{\sim}{c} > \underset{\sim}{0}$.

Lemma 3.6. For all $e > \omega$: If $D = \{e\}^{A \oplus B}$ then every attacker of a requirement $R_{\langle e,i\rangle}$, $i < \omega$, is eventually enumerated in E_e or cancelled.

Proof. Assume that y is an attacker of the requirement $R_{\langle e,i\rangle}$ which is neither cancelled nor enumerated in E_e. Let s+1 be the stage at which y is appointed. Then there exists $x < \omega$, such that $D_{s+1}(x) = 1$, $\{e\}_s^{A_s \oplus B_s}(x) = 0$, $y \geq u$ where $u = u(e, A_s \oplus B_s, x, s)$, and $r(\langle e,i\rangle, s+1) = u$. Since y is never cancelled,

$\forall t \geq s+1$ ($r(\langle e,i\rangle, t) = r(\langle e,i\rangle, s+1) = u$)

and thus $B_s[u] = B_{s+1}[u] = B[u]$. Further, since y is never enumerated in E_e, $A_s[y] = A[y]$. It follows that the computation $\{e\}_s^{A_s \oplus B_s}(x)$ is $A \oplus B$ - correct, i.e. $\{e\}^{A \oplus B}(x) = 0$, and thus $D \neq \{e\}^{A \oplus B}$.

Lemma 3.7. For all $e<\omega$,

(a) $E_e \leq_T A$

(b) $D = \{e\}^{A \oplus B} \implies E_e \leq_T C$.

Proof. (a) : By permitting.

(b) : Let $D = \{e\}^{A \oplus B}$. Notice that only attackers of requirements $R_{\langle e,i\rangle}$, $i<\omega$, are enumerated in E_e. Further an attacker y is appointed only at a stage s+1 at which it is permitted by C (i.e. $\exists z< y\ (\ z\in C_{s+1} \setminus C_s\ )$).

Hence $E_e \leq_T C$ is seen as follows . Given $x<\omega$, to decide if $x\in E_e$, C-recursively find a stage s_0 such that $C_{s_0}[x] = C[x]$. By lemma 3.6 (effectively) find a stage $s_1 \geqslant s_0$ s.t. all attackers of requirements $R_{\langle e,i\rangle}$, $i<\omega$, which were appointed before stage s_0 are cancelled or in E_e by the end of stage s_1. Then $x\in E_e$ iff $x\in E_{e,s_1}$.

It follows from lemmata 3.3 and 3.4 that B is effectively non - cappable. Lemmata 3.5 and 3.7 imply that $A \oplus B$ is incomplete. (Assume not. Then $D = \{e\}^{A \oplus B}$, for some e, since D is r.e.. Then, by lemma 3.5, E_e nonrecursive. Since by lemma 3.7 $E_e \leq_T A,C$, this contradicts $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$.)

Remark. Though $dg(A \oplus B)$ constructed above is effectively noncappable, the proof of theorem 3.1 doesn't show that

(3.21) $\forall \underset{\sim}{a} < \underset{\sim}{0}' \exists \underset{\sim}{b} \in ENC\ (\ \underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{0}')$.

The proof can be modified, however, to get (3.21) and the stronger result

(3.22) $\exists \underset{\sim}{a} \in ENC\ \forall \underset{\sim}{b}\ (\ \underset{\sim}{b} \notin ENC \implies \underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}')$.

Since the main theorem of the next section will imply (3.21) and (3.22) (cf. Corollary 4.2), we omit a direct proof here.

§ 4 AN EXTENSION OF LACHLAN'S NON-DIAMOND THEOREM.

Lachlan proved in [4] that there is no embedding of the diamond, i.e. the four-element Boolean algebra, into the r.e. degrees which preserves both greatest and least elements. In other words, for any incomplete r.e. degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$,

(4.1) $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$ or $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1 \neq \underset{\sim}{0}$.

He also observed that, by relativization, (4.1) can be extended to

(4.2) $\underset{\sim}{b}$ low $\Rightarrow$ ($\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$ or $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1 \neq \underset{\sim}{b}$)

for any r.e. degrees $\underset{\sim}{a}_0, \underset{\sim}{a}_1 \neq \underset{\sim}{0}'$ and $\underset{\sim}{b}$, and used this to prove that the r.e. degrees don't form a lattice.

However, Shoenfield and Soare [19], and independently Lachlan [7], showed that the condition "$\underset{\sim}{b}$ low" in (4.2) can not be weakend to "$\underset{\sim}{b}$ incomplete" : There is an embedding of the diamond into $\underset{\sim}{R}$ which preserves the greatest element.

Jockusch (cf. Soare [29],§6) proved a further extension of (4.1) in an other direction (using esentially Lachlan's method): For any $n \geqslant 0$ and any r.e. dgrees $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_n$,

(4.3) $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or $\exists i \leq n$ ($\underset{\sim}{a}_i$ is noncappable).

(4.3) implies that the cappable degrees don't generate $\underset{\sim}{R}$.

For the special case n = 1, (4.3) says that if two incomplete r.e. degrees join to $\underset{\sim}{0}'$ then at least one of them is noncappable. Harrington's cup-and-cap theorem (unpublished) states, however, that not both of them need be noncappable; i.e.

(4.4) $\exists \underset{\sim}{a}_0, \underset{\sim}{a}_1, \underset{\sim}{a}_2$ ($\underset{\sim}{a}_0 < \underset{\sim}{0}'$, $\underset{\sim}{a}_2 > \underset{\sim}{0}$, $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$ and $\underset{\sim}{a}_1 \cap \underset{\sim}{a}_2 = \underset{\sim}{0}$).

Again by relativization, (4.3) can be extended to

(4.5) ($\underset{\sim}{b}$ low and $\underset{\sim}{b} \leq \underset{\sim}{a}_0, \ldots \underset{\sim}{a}_n$) $\Longrightarrow$

$\Longrightarrow$ ($\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or $\exists i \leq n \, \forall \underset{\sim}{c} > \underset{\sim}{b}$ ($\underset{\sim}{a}_i \cap \underset{\sim}{c} \neq \underset{\sim}{b}$))

where $n \geq 0$ and $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_n$ and $\underset{\sim}{b}$ are any r.e. degrees.
We show that (4.5) can be further generalized as follows : For any $n \geq 0$ and any r.e. degrees $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_n, \underset{\sim}{b}_0, \ldots, \underset{\sim}{b}_n$,

(4.6) $\underset{\sim}{b}_0 \cup \ldots \cup \underset{\sim}{b}_n$ low $\Rightarrow$ ($\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or
$\exists i \leq n \forall \underset{\sim}{c} \not\leq \underset{\sim}{b}_i \exists \underset{\sim}{d}$ ($\underset{\sim}{d} \leq \underset{\sim}{a}_i, \underset{\sim}{c}$ and $\underset{\sim}{d} \not\leq \underset{\sim}{b}_i$)).

Indeed we will prove an effective version of (4.6).
An interesting special case of (4.6) is

(4.7) $\forall \underset{\sim}{a}_0, \underset{\sim}{a}_1 ($ ($\underset{\sim}{a}_0$ low and $\underset{\sim}{a}_1$ cappable) $\Rightarrow$ $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$).

This shows that (4.4) fails if "$\underset{\sim}{a}_0 < \underset{\sim}{0}'$" is replaced by the stronger condition "$\underset{\sim}{a}_0$ low". Another consequence of (4.7) is

(4.8) $\exists \underset{\sim}{a} \in NC \, \forall \underset{\sim}{b}$ ($\underset{\sim}{b}$ cappable $\Rightarrow$ $\underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}'$),

which implies theorem 3.1. Using the effective version of (4.6) we get theorem 3.1 for effectively noncappable degrees instead of noncappable degrees.
We also get a partial answer to the following question of Jockusch: Is it true that for every r.e. degree $\underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}'$ there is an r.e. degree $\underset{\sim}{b}$ s.t. $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist? By (4.7) and the splitting theorem, this is true for all low degrees $\underset{\sim}{a}$.
Another similar consequence is that there exists an r.e. degree $\underset{\sim}{a}$ ($\neq \underset{\sim}{0}, \underset{\sim}{0}'$) s.t. , for all r.e. degrees $\underset{\sim}{b} \mid \underset{\sim}{a}$, $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist or is not low.(Hence if $\underset{\sim}{b}$ is low then $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist .)
In § 5 we will extend the last two results using a more direct approach.
The proof of (4.6) combines a new variant of Lachlan's proof of (4.1) with Robinson's technique for handling computations with a low oracle set (cf. [14]). To make the proof more transparent, we first prove the special case $n = 1$ of (4.6) (Theorem 4.1) and then modify this proof to get the general result (Theorem 4.2).

We start with a definition needed to state the theorems properly :

Definition 4.1. (a) An r.e. degree $\underset{\sim}{a}$ is $\underset{\sim}{b}$-cappable if

$$\exists \underset{\sim}{c} \not\leq \underset{\sim}{b}\ \forall \underset{\sim}{d}\ (\ \underset{\sim}{d} \leq \underset{\sim}{a}, \underset{\sim}{c} \Rightarrow \underset{\sim}{d} \leq \underset{\sim}{b}\ ).$$

Otherwise $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable.

(b) An r.e. degree $\underset{\sim}{a}$ is effectively non-$\underset{\sim}{b}$-cappable if there exists a sequence of uniformly r.e. sets A_e, $e < \omega$, s.t. for any $A \in \underset{\sim}{a}$ and $B \in \underset{\sim}{b}$

(i) $\forall e\ (\ A_e \leq_T A$ uniformly in e),

(ii) $\forall e\ (\ A_e \leq_T W_e$ uniformly in e) and

(iii) $\forall e\ (\ W_e \not\leq_T B \Rightarrow A_e \not\leq_T B\ )$.

Let $NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$) be the set of the (effectively) non-$\underset{\sim}{b}$-cappable degrees.

We will need the following facts on (effectively) non-$\underset{\sim}{b}$-cappable degrees.

Proposition 4.1. For all $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$:

(a) $\underset{\sim}{a} \neq \underset{\sim}{0}' \Rightarrow \underset{\sim}{a}$ is $\underset{\sim}{a}$-cappable

(b) ($\underset{\sim}{c} < \underset{\sim}{a}$ and $\underset{\sim}{a}$ $\underset{\sim}{b}$-cappable) $\Rightarrow$ $\underset{\sim}{c}$ $\underset{\sim}{b}$-cappable

(c) The set $NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$) is a filter in $\underset{\sim}{R}$; i.e.

(c1) ($\underset{\sim}{a} < \underset{\sim}{c}$ and $\underset{\sim}{a} \in NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$)) $\Rightarrow$ $\underset{\sim}{c} \in NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$)

(c2) ($\underset{\sim}{a}, \underset{\sim}{c} \in NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$) and $\underset{\sim}{a} \cap \underset{\sim}{c}$ exists) $\Rightarrow$

$\Rightarrow$ $\underset{\sim}{a} \cap \underset{\sim}{c} \in NC^{\underset{\sim}{b}}$ ($ENC^{\underset{\sim}{b}}$)

(d) $ENC^{\underset{\sim}{b}} \subseteq NC^{\underset{\sim}{b}}$

(e) $NC^{\underset{\sim}{0}} = NC$ and $ENC^{\underset{\sim}{0}} = ENC$.

Proof. The proofs of (a), (b), (d) and (e) are trivial. (c) is

proved similarly to propositions 1.1 and 2.2.

Theorem 4.1. If $\underset{\sim}{a}_0$, $\underset{\sim}{a}_1$, $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ are r.e. degrees such that $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ is low then either $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$ or $\underset{\sim}{a}_i \in NC^{\underset{\sim}{b}_i}$ for some $i \leq 1$.

Proof. Let any r.e. degrees $\underset{\sim}{a}_0$, $\underset{\sim}{a}_1$, $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$, s.t. $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ is low, be given. Further fix any r.e. degrees $\underset{\sim}{c}_0$ and $\underset{\sim}{c}_1$ such that $\underset{\sim}{c}_i \not\leq \underset{\sim}{b}_i$ (i=0,1). It suffices to show that one of the following conditions holds :

(4.10) $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$

(4.11) $\exists \underset{\sim}{e}$ ($\underset{\sim}{e} \leq \underset{\sim}{a}_0, \underset{\sim}{c}_0$ and $\underset{\sim}{e} \not\leq \underset{\sim}{b}_0$)

(4.12) $\exists \underset{\sim}{e}$ ($\underset{\sim}{e} \leq \underset{\sim}{a}_1, \underset{\sim}{c}_1$ and $\underset{\sim}{e} \not\leq \underset{\sim}{b}_1$).

Choose r.e. sets $A_i \in \underset{\sim}{a}_i$, $B_i \in \underset{\sim}{b}_i$ and $C_i \in \underset{\sim}{c}_i$ s.t. A_0 and B_0 consist of even numbers only and A_1 and B_1 consist of odd numbers only ($i \leq 1$). Fix effective enumerations $\{A_{i,s} : s < \omega\}$, $\{B_{i,s} : s < \omega\}$ and $\{C_{i,s} : s < \omega\}$ of A_i, B_i and C_i, respectively ($i \leq 1$). Set $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. Obviously, $dg(A) = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $dg(B) = \underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$. $\{A_s : s < \omega\}$ and $\{B_s : s < \omega\}$ are effective enumerations of A and B, respectively, where $A_s = A_{0,s} \cup A_{1,s}$ and $B_s = B_{0,s} \cup B_{1,s}$.

A set D is a witness for (4.10) if D is Δ^0_2 and $D \not\leq_T A$. Since every Δ^0_2-set is recursive in any complete set, the existence of a witness for (4.10) implies that (4.10) holds.

A set E is a witness for (4.11) ((4.12)) if E is r.e. and, for i=0 (i=1) $E \leq A_i, C_i$ and $E \not\leq_T B_i$.

To show that one of the conditions (4.10) - (4.12) holds we will simultaneously construct sets which are intended as potential witnesses for the individual conditions. While we will construct

a single set D for (4.10), it will be convenient to construct a sequence $\{E_e : e<\omega\}$ of potential witnesses for (4.11) and a double sequence $\{E_{(e,i)} : e,i<\omega\}$ of potential witnesses for (4.12). The set D will be Δ_2^0 and E_e and $E_{(e,i)}$ will be r.e. sets s.t. $E_e \leq_T A_0$ and $E_{(e,i)} \leq_T A_1$ (by permitting). Hence D will witness (4.10) if we meet the requirements

$$\hat{R}_e \quad : \quad D \neq \{e\}^A$$

for all $e<\omega$. To make E_e witness (4.11) we have to insure that $E_e \leq_T C_0$ and meet the requirements

$$\hat{R}_i^e \quad : \quad E_e \neq \{i\}^{B_0}$$

for all $i<\omega$. Finally to make $E_{(e,i)}$ witness (4.12) we must insure $E_{(e,i)} \leq_T C_1$ and meet the requirements

$$\hat{R}_j^{(e,i)} \quad : \quad E_{(e,i)} \neq \{j\}^{B_1}$$

for all $j<\omega$.

We will show that if D fails to witness (4.10) and a requirement, say $\hat{R}_e$, is not met then $E_e \leq_T C_0$. Hence E_e only fails to witness (4.11) if, for some i, $\hat{R}_i^e$ is not met. In that case we show that $E_{(e,i)} \leq_T C_1$ and all requirements $R_j^{(e,i)}$, where $j<\omega$, are met, i.e. $E_{(e,i)}$ is a witness for (4.12).

This will be done by satisfying the requirements

$$R_e \quad : \quad D \neq \{e_0\}^A \text{ or } E_{e_0} \neq \{e_1\}^{B_0} \text{ or } E_{(e_0,e_1)} \neq \{e_2\}^{B_1}$$

and the conditions

$$(4.13) \quad D = \{e_0\}^A \Longrightarrow E_{e_0} \leq_T C_0$$

and

$$(4.14) \quad (D = \{e_0\}^A \text{ and } E_{e_0} = \{e_1\}^{B_0}) \Longrightarrow E_{(e_0,e_1)} \leq_T C_1$$

for all $e,e_0,e_1,e_2 < \omega$ s.t. $e = \langle e_0,e_1,e_2\rangle$.

Since B is low, we can fix a recursive function f such that

$$(4.15) \quad \forall e \, (W_e \cap \{x : \mathbb{D}_x \subseteq \bar{B}\} \subseteq W_{f(e)})$$

and

(4.16) $\forall e \, (\, W_e \cap \{ x : \mathbb{D}_x \subseteq \overline{B} \} = \emptyset \implies W_{f(e)} \text{ finite })$

(cf. Soare [27], Theorem 2.7).

The function f will be used to guess the B_i-correctness of the computations $\{e\}^{B_i}_s(x)$ ($e,x,s < \omega, i \leq 1$). These guesses will also use an r.e. test set V which will be given by the construction. To apply (4.15) and (4.16), an index of V must be taken to be known in advance. Therefore we simultaneously do infinitely many constructions, one for each $n < \omega$. In the n-th construction we assume that n is the index of the set V (enumerated in this construction). This will be done uniformly in n, so that by the recursion theorem for some n the assumption about the index of V is correct. We will show that the construction for this n has the desired properties.

Let D_s consist of those numbers which are put into D at some stage $\leq s$ and are not extracted from D at a later stage $\leq s$.

Further let $\lambda x,y([x]_y)$ be a recursive function such that, for all $x,y < \omega$, $W_{[x]_y} = \{ z : \langle y,z \rangle \in W_x \}$.

We can now describe the n-construction ($n < \omega$):

Stage s=0. Do nothing.

Stage s+1. For all $e = \langle e_0,e_1,e_2 \rangle \leq s+1$ perform the following steps.

Step 1. If there exists no attack on R_e at the end of stage s and if there exist numbers u and x such that

$$\{e_0\}^{A_s}_s(e) = 0,$$

$$\mathbb{D}_u = \{ z : z < u(e_0,A_s,e,s) \text{ and } z \notin A_s \},$$

$$x \geq u,$$

$$(4.17)\quad \begin{cases} x = \langle e, x' \rangle \text{ for some } x' < \omega, \\ \{e_2\}_s^{B_{1,s}}(x) = 0, \text{ and} \\ \exists y < x \ (y \in C_{1,s} \setminus C_{1,s-1}), \end{cases}$$

then start an attack on R_e. Let x_1 be the least x which satisfies (4.17) and let v_1 be defined by

$\mathbb{D}_{v_1} = \{ z : z < u(e_2, B_{1,s}, x_1, s),\ z \notin B_{1,s} \text{ and } z \text{ odd} \}$. x_1 is appointed as B_1-attacker and u and v_1 are the A- and B-information, respectively, of the attack.

Step 2. If there exists an attack on R_e at the end of stage s, which is not yet advanced and whose A-information is u, and if $A_s \cap \mathbb{D}_u = \emptyset$ and there exists some x which satisfies (4.17) and is smaller than the B_1-attacker of the attack on R_e, then let x_1 be the least number x which satisfies (4.17), let v_1 be defined by

$\mathbb{D}_{v_1} = \{ z : z < u(e_2, B_{1,s}, x_1, s),\ z \notin B_{1,s} \text{ and } z \text{ odd} \}$;

replace the old B_1-attacker by x_1 and replace the old B-information by v_1.

Step 3. If at the end of stage s there exists an attack on R_e, which is not yet advanced, whose A- and B-information is u and v_1, respectively, and whose B_1-attacker was not replaced in step 2 of this stage, and if further :

$A_s \cap \mathbb{D}_u = B_s \cap \mathbb{D}_{v_1} = \emptyset$

and there exists some x such that

$$(4.18)\quad \begin{cases} x > u, \\ x = \langle e, x' \rangle \text{ for some } x', \\ \{e_1\}_s^{B_{0,s}}(x) = 0, \text{ and} \\ \exists y < x \ (y \in C_{0,s} \setminus C_{0,s-1}), \end{cases}$$

then let x_0 be the least x which satisfies (4.18) and let v be

defined by

$$\mathbb{D}_v = \mathbb{D}_{v_1} \cup \{ z : z < u(e_1, B_{0,s}, x_0, s);\ z \notin B_{0,s} \text{ and } z \text{ even} \}.$$

Put $\langle e,v \rangle$ into V and simultaneously enumerate $W_{f([n]_e)}$ and B until v shows up in $W_{f([n]_e)}$ or an element of $\mathbb{D}_v$ shows up in B (if none of both happens, stage s+1 is never completed). In the first case advance the attack : Put e into D, appoint x_0 as B_0-attacker of the attack, and replace the old B-information by v.

Step 4. If there exists an advanced but not yet completed attack on R_e at the end of stage s such that the attack's A- and B-information are u and v, respectively, and

$$(A_s \setminus A_{s-1}) \cap \mathbb{D}_u \neq \emptyset \quad \text{and} \quad B_s \cap \mathbb{D}_v = \emptyset,$$

then complete the attack: If the least element of $(A_s \setminus A_{s-1}) \cap \cap \mathbb{D}_u$ is even, put the B_0-attacker of the attack into E_{e_0}. Otherwise put the B_1-attacker into $E_{(e_0,e_1)}$.

Step 5. If there exists an attack on R_e at the end of stage s such that u and v are the A- and B-information(of the attack), respectively, and either the B_1-attacker of the attack is not replaced in step 2 at this stage and $B_s \cap \mathbb{D}_v \neq \emptyset$ or the attack is not yet advanced and $A_s \cap \mathbb{D}_u \neq \emptyset$, then cancel the attack and if $e \in D_s$ extract e from D.

End of construction.

Note that the n-construction is effective uniformly in n, $n < \omega$. Hence the sets V_n are uniformly r.e., where V_n denote the set V built in the n-construction. Therefore, by the recursion theorem, there exists an n such that $V_n = W_n$. We fix such an n for the following and we show that the n-construction for this n has the desired properties.

Lemma 4.1. Every stage of the construction is completed after finitely many steps.

Proof. The only critical point is in step 3 of the construction, where we put a certain number $\langle e,v\rangle$ into V and simultaneously enumerate $W_{f([n]_e)}$ and B until either v shows up in $W_{f([n]_e)}$ or an element of $\mathbb{D}_v$ shows up in B. We have to show that one of both happens and thus the enumeration process terminates. Since $V = W_n$, and thus $v \in W_{[n]_e}$, this is true by (4.15).

Lemma 4.2. E_{e_0} and $E_{(e_0,e_1)}$ are r.e. , $E_{e_0} \leq_T A_0$ and $E_{(e_0,e_1)} \leq_T A_1$ (for all $e_0,e_1 < \omega$).

Proof. By the efectiveness of the construction, E_{e_0} and $E_{(e_0,e_1)}$ are r.e.. $E_{e_0} \leq_T A_0$ and $E_{(e_0,e_1)} \leq_T A_1$ hold by permitting. (E.g. a number x enters E_{e_0} at stage s+1 only if there is an attack on a requirement such that x is the B_0-attacker of the attack, the attack is completed at stage s+1 and, if u is the A-information of the attack, an element y of $\mathbb{D}_u$ enters A_0 at stage s. Since $y < u$ and, by construction, $u \leq x$, it follows that

$$x \in E_{e_0,s+1} \setminus E_{e_0,s} \Longrightarrow \exists y < x \ (\ y \in A_{0,s} \setminus A_{0,s-1} \ ).)$$

The A-information u of an attack is correct (at stage s+1) if $A \cap \mathbb{D}_u = \emptyset$ ($A_s \cap \mathbb{D}_u = \emptyset$). Analogously, the B-information v is correct (at stage s+1) if $B \cap \mathbb{D}_v = \emptyset$ ($B_s \cap \mathbb{D}_v = \emptyset$). The B-information of an advanced attack is called final. Note that the A-information of an uncompleted attack at the end of a stage s+1 is correct at stage s+1. The B-information of any attack at the end of stage s+1 is correct at stage s+1. If the B-information of an attack at some stage is final, then the attack has the same B-information at all following stages (as long the

attack is not cancelled). The A-information of an attack is the same at all stages.

Lemma 4.3. Only finitely many attacks on R_e are advanced (for all $e < \omega$).

Proof. If there exists an attack on R_e which is never cancelled, then the claim is obvious (Note that at every stage at most one attack on R_e exists.). Hence we may assume that all attacks on R_e are eventually cancelled. Note that an advanced attack is only cancelled if its final B-information is not correct. This implies that $W_{[n]_e} \cap \{x : \mathbb{D}_x \subseteq \overline{B}\} = \{x : \langle e,x\rangle \in V \wedge \mathbb{D}_x \subseteq \overline{B}\} = \emptyset$ and hence, by (4.16), that $W_{f([n]_e)}$ is finite.
Further note that the final B-informations for different attacks on R_e are different: If v is the final B-information of an earlier attack, and v' is the final B-information of a later attack, and s+1 is the stage where the later attack is advanced, then v is not correct at stage s+1, while v' is correct at stage s+1.
Since $W_{f([n]_e)}$ contains all final B-informations of attacks on R_e, it follows that only finitely many attacks on R_e are advanced.

Lemma 4.4. D is a Δ^0_2-set.

Proof. A number x is put into D only at stages where an attack on R_x is advanced. By lemma 4.3 there are only finitely many such stages. Hence, for all $x < \omega$,

$x \in D$ iff $\exists y \forall z \geq y\ (x \in D_z)$ iff $\forall y \exists z \geq y\ (x \in D_z)$.

Since the binary relation $x \in D_z$ is recursive (by the effectiveness of the construction), this implies that D is a Δ^0_2-set.

Lemma 4.5. If $D = \{e_0\}^A$ and $e = \langle e_0, e_1, e_2\rangle$ then every advanced attack on R_e will eventually be completed or cancelled

(for all $e, e_0, e_1, e_2 < \omega$).

Proof. Assume for a contradiction that $D = \{e_0\}^A$, $e = \langle e_0, e_1, e_2 \rangle$ and there exists an advanced attack on R_e which is neither completed nor cancelled (at any stage). Fix such an attack, let u be the A-information of the attack, and let s+1 be the stage where the attack is advanced. Then, as it is easily seen, $D_{s+1}(e) = 1$, $\{e_0\}^A_s(e) = 0$, and $\mathbb{D}_u = \{ z : z < u(e_0, A_s, e, s) \text{ and } z \notin A_s \}$. Since the attack is never completed or cancelled, e isn't extracted from D after stage s+1 and the A-information u is correct. It follows that $D(e) = 1 \neq 0 = \{e_0\}^A(e)$ contrary to the assumption that $D = \{e_0\}^A$.

Lemma 4.6. Let $D = \{e_0\}^A$, $E_{e_0} = \{e_1\}^{B_0}$ and $e = \langle e_0, e_1, e_2 \rangle$. Then every attack on R_e is eventually completed or cancelled.

Proof. Assume for a contradiction that there exists an attack on R_e which is never completed or cancelled. By lemma 4.5 this attack is never advanced. Note that the B_1-attacker of an attack is replaced only finitely often, since B_1-attackers are replaced by smaller numbers only. Hence there exist s_0, x_1, u and v_1 such that at all stages $s \geq s_0$ x_1 is the B_1-attacker and u and v_1 are the A- and B-information, respectively, of the attack. Then

(+) $\mathbb{D}_u \cap A = \mathbb{D}_{v_1} \cap B = \emptyset$,

since otherwise the attack would be cancelled.

Since no attack on R_e is advanced after stage s_0 (and by our convention : $\{e_1\}^{B_0}_{s}(x)\downarrow \implies x < s$), $E_{e_0}(\langle e, x \rangle) = 0$ for all $x \geq s_0$. By the assumption that $E_{e_0} = \{e_1\}^{B_0}$, there exists a B_0-recursive function g such that for all $x < \omega$, $g(x) \geq s_0$, and

(++) $\forall y \leq \langle e,x\rangle$ ($\{e_1\}_{g(x)}^{B_0,g(x)}(y)\downarrow$ via a B_0-correct computation).

Hence, since $C_0 \not\leq_T B_0$, there must exist numbers x' and s such that $x' \geq \langle e,s_0\rangle, u$, $s \geq g(x')$, and $\exists y < x'(y \in C_{0,s} \setminus C_{0,s-1})$. Fix such x' and s, and let $x = \langle e,x'\rangle$. Then x satisfies (4.18) at stage $s+1$. Hence, by (+), in step 3 of stage $s+1$ numbers x_0 and v are chosen s.t. $x_0 \leq x$ is the least number which satisfies (4.18) and $\mathbb{D}_v = \mathbb{D}_{v_1} \cup \{ z : z < u(e_1,B_{0,s},x_0,s),\ z \notin B_{0,s}$ and z is even $\}$. $\langle e,v\rangle$ is put into V, and $W_{f([n]_e)}$ and B are simultaneously enumerated until either v shows up in $W_{f([n]_e)}$ or an element of $\mathbb{D}_v$ shows up in B. Since, by (+) and (++), $\mathbb{D}_v \cap B = \emptyset$, the first must happen, and thus the attack on R_e is advanced at stage $s+1$. We saw above, though, that the attack is never advanced. Contradiction.

Lemma 4.7. If $D = \{e_0\}^A$ then $E_{e_0} \leq_T C_0$ (for all $e_0 < \omega$).

Proof. Let $D = \{e_0\}^A$. By lemma 4.5 there exists a recursive function g such that, for all $s < \omega$, every attack on a requirement $R_{\langle e_0,e_1,e_2\rangle}$, where $e_1,e_2 < \omega$, which is advanced before stage $s+1$ is completed or cancelled before stage $g(s+1)$.

Note that a number x is enumerated into E_{e_0} at stage $s+1$ only if x is the B_0-attacker of an attack on a requirement R_e, where $e = \langle e_0,e_1,e_2\rangle$ for some $e_1,e_2 < \omega$, which is completed at stage $s+1$ and which was advanced at a stage $t+1$ such that at stage t a number less then x entered C_0.

It follows that

$$C_{0,s}[x+1] = C_0[x+1] \implies E_{e_0,g(s)}[x+1] = E_{e_0}[x+1]$$

for all $s,x < \omega$. Hence $E_{e_0} \leq_T C_0$.

Lemma 4.8. If $D = \{e_0\}^A$ and $E_{e_0} = \{e_1\}^{B_0}$ then $E_{(e_0,e_1)} \leq_T C_1$ (for all $e_0, e_1 < \omega$).

Lemma 4.8 follows from lemma 4.6 similarly to the way lemma 4.7 follows from lemma 4.5.

Lemma 4.9. For all $e < \omega$, the requirement R_e is met.

Proof. Let $e = \langle e_0, e_1, e_2 \rangle$ be given and assume for a contradiction that R_e is not met, i.e. that $D = \{e_0\}^A$, $E_{e_0} = \{e_1\}^{B_0}$ and $E_{(e_0,e_1)} = \{e_2\}^{B_1}$.

Note that if there is a completed attack on R_e which is never cancelled, then $E_{e_0}(x_0) = 1 \neq 0 = \{e_1\}^{B_0}(x_0)$ or $E_{(e_0,e_1)}(x_1) = 1 \neq 0 = \{e_2\}^{B_1}(x_1)$, where x_0 and x_1 are the B_0- and B_1-attacker of the attack, respectively.

Hence we may assume that every completed attack on R_e is cancelled. It follows, by lemma 4.6, that every attack on R_e is eventually cancelled, Hence $D(e) = 0$. By the assumption that $D = \{e_0\}^A$, we can choose numbers u and s_0 such that

$$\forall s \geq s_0 \; (\{e_0\}^{A_s}_s(e) = 0 \text{ and } \mathbb{D}_u = \{ z : z < u(e_0, A_s, e, s) \text{ and } z \notin A_s \}).$$

By lemma 4.2 we may assume that no attack on R_e after stage s_0 is ever advanced. Note that every attack on R_e after stage s_0 has the correct A-information u.

Since no attack on R_e is advanced after stage s_0, for all $x \geq s_0$, $E_{(e_0,e_1)}(\langle e,x \rangle) = 0$, and thus, by the assumption that $E_{(e_0,e_1)} = \{e_2\}^{B_1}$, $\{e_2\}^{B_1}(\langle e,x \rangle) = 0$. By the same assumption there is a B_1-recursive function g such that, for all $x < \omega$, $g(x) \geq s_0$ and

(+) $\forall y \leq \langle e,x \rangle \; (\{e_2\}^{B_1,g(x)}_{g(x)}(y) \downarrow$ and the computation is B_1-correct).

Hence, since $C_1 \not\leq_T B_1$, there must exist x and s such that $x > s_0, u$,

$s \geq g(x)$ and $\exists y < x\ (\ y \in C_{1,s} \setminus C_{1,s-1}\ )$.

If there is no attack on R_e at the end of stage s, an attack on R_e is started at stage s+1 and a B_1-attacker $\leq \langle e,x \rangle$ is appointed. If there is an attack on R_e at the end of stage s whose B_1-attacker is greater than $\langle e,x \rangle$, the old attacker is replaced by a number $\leq \langle e,x \rangle$. Hence, in any case, there exists an attack on R_e at the end of stage s or s+1 whose B_1-attacker is less than or equal to $\langle e,x \rangle$. Since the attack is never advanced, the B_1-attacker can be replaced by a smaller number only. Hence (+) implies that after stage s the attack has always a correct B -information. Since the A-information of the attack is u, which is correct too, the attack is never cancelled. This contradicts our earlier observation that every attack on R_e is cancelled.

As we have seen earlier, the theorem follows from lemmata 4.2, 4.4, 4.7, 4.8 and 4.9.

We now state the general theorem.

Theorem 4.2. Let $n < \omega$ and let $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$, $1 \leq i \leq n$, be r.e. degrees such that $\underset{\sim}{b}_1 \cup \ldots \cup \underset{\sim}{b}_n$ is low. Then either $\underset{\sim}{a}_1 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or for some i, where $1 \leq i \leq n$, $\underset{\sim}{a}_i \in \mathrm{ENC}^{\underset{\sim}{b}_i}$.

Corollary 4.1. If $\underset{\sim}{c}_1, \ldots \underset{\sim}{c}_n$ ($n < \omega$) are r.e. degrees such that $\underset{\sim}{c}_1$ is low and $\underset{\sim}{c}_2, \ldots, \underset{\sim}{c}_n \notin \mathrm{ENC}$ then $\underset{\sim}{c}_1 \cup \ldots \cup \underset{\sim}{c}_n < \underset{\sim}{0}'$.

Proof. Choose r.e. degrees $\underset{\sim}{c}_1, \ldots \underset{\sim}{c}_n$ s.t. $\underset{\sim}{c}_1$ is low and $\underset{\sim}{c}_2, \ldots, \underset{\sim}{c}_n \notin \mathrm{ENC}$. Then the theorem yields, for $\underset{\sim}{a}_i = \underset{\sim}{c}_i$ ($1 \leq i \leq n$), $\underset{\sim}{b}_1 = \underset{\sim}{c}_1$ and $\underset{\sim}{b}_j = \underset{\sim}{0}$ ($2 \leq j \leq n$), that $\underset{\sim}{c}_1 \cup \ldots \cup \underset{\sim}{c}_n < \underset{\sim}{0}'$ or $\underset{\sim}{c}_1 \in \mathrm{ENC}^{\underset{\sim}{c}_1}$ or

for some j, $2 \leq j \leq n$, $\underset{\sim}{c}_j \in \mathrm{ENC}^{\underset{\sim}{0}} = \mathrm{ENC}$. Since by proposition 4.1 $\underset{\sim}{c}_1 \notin \mathrm{ENC}^{\underset{\sim}{c}_1}$, $\underset{\sim}{c}_1 \cup \ldots \cup \underset{\sim}{c}_n < \underset{\sim}{0}'$ must hold.

Note that corollary 4.1 applies if $\underset{\sim}{c}_2, \ldots \underset{\sim}{c}_n$ are cappable.

Corollary 4.1 implies an effective form of Jockusch's extension (4.3) of the non-diamond theorem : For any r.e. degrees $\underset{\sim}{a}_0, \ldots \underset{\sim}{a}_n$

(4.20) $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or $\exists i \leq n$ ($\underset{\sim}{a}_i$ effectively noncappable).

The corollary can also be used to show that some constructions automatically yield (effectively) noncappable degrees. E.g. the first examples of r.e. degrees $\neq \underset{\sim}{0}, \underset{\sim}{0}'$, the incomparable degrees constructed by Friedberg [2] and Muchnik [13] are both (effectively) noncappable. This follows from Corollary 4.1 since Soare [25] has shown that these degrees are both low and that they join to $\underset{\sim}{0}'$.

Corollary 4.1 also implies the following result on (effectively) noncappable degrees, which extends theorem 3.1.

Corollary 4.2.

(a) $\exists \underset{\sim}{a} \in \mathrm{ENC}\ \forall \underset{\sim}{b}$ ($\underset{\sim}{b} \notin \mathrm{ENC} \Rightarrow \underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}'$).

(b) $\forall \underset{\sim}{b} < \underset{\sim}{0}'\ \exists \underset{\sim}{c}$ ($\underset{\sim}{c} \in \mathrm{ENC}$ and $\underset{\sim}{b} < \underset{\sim}{c} < \underset{\sim}{0}'$).

Proof. (a): By corollary 4.1 any low degree $\underset{\sim}{a} \in \mathrm{ENC}$ will do. The existence of such a degree was shown in § 2 . (By the previous remark the Friedberg - Muchnik degrees also have this property).

(b): Let $\underset{\sim}{b} < \underset{\sim}{0}'$ be given. If $\underset{\sim}{b} \in \mathrm{ENC}$, then (b) follows from the density theorem and the fact that ENC is closed upwards in $\underset{\sim}{R}$ (cf. proposition 2.2). If $\underset{\sim}{b} \notin \mathrm{ENC}$, then by (a) we can choose a degree $\underset{\sim}{a} \in \mathrm{ENC}$ s.t. $\underset{\sim}{a} \cup \underset{\sim}{b} < \underset{\sim}{0}'$. Again, since ENC is closed up-

wards, $\underset{\sim}{a} \cup \underset{\sim}{b} \in ENC$.

Corollary 4.3. If $\underset{\sim}{a} \neq \underset{\sim}{0}$ is low, then there exists some (low) r.e. degree $\underset{\sim}{b}$ such that $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist.

Proof. Let $\underset{\sim}{a}$ be a nonzero low degree. By the spliiting theorem, there exist (low) r.e. degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$ and $\underset{\sim}{a} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$. We claim that $\underset{\sim}{a} \cap \underset{\sim}{a}_0$ or $\underset{\sim}{a} \cap \underset{\sim}{a}_1$ doesn't exist. Otherwise, for $i \leq 1$, $\underset{\sim}{a}_i$ would be $(\underset{\sim}{a} \cap \underset{\sim}{a}_i)$-cappable and, since $\underset{\sim}{a}$ is low, $(\underset{\sim}{a} \cap \underset{\sim}{a}_0) \cup (\underset{\sim}{a} \cap \underset{\sim}{a}_1)$ would be low. This would contradict theorem 4.1.

Corollary 4.4.

(a) $\exists \underset{\sim}{a} < \underset{\sim}{0}' \forall \underset{\sim}{b}$ ($\underset{\sim}{b}$ low $\Longrightarrow$ $\underset{\sim}{a} \in ENC^{\underset{\sim}{b}}$)

(b) $\exists \underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}' \forall \underset{\sim}{b}$ (($\underset{\sim}{b}$ low and $\underset{\sim}{a} \mid \underset{\sim}{b}$) $\Rightarrow$ $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist).

Proof. (a): By Harrington's cup-and-cap theorem (4.4), there exist degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$, $\underset{\sim}{a}_0 < \underset{\sim}{0}'$ and $\underset{\sim}{a}_1$ is cappable. By theorem 4.2 $\underset{\sim}{a}_0 \in ENC^{\underset{\sim}{b}}$ for all low degrees $\underset{\sim}{b}$.
(b) follows from (a).

Proof of theorem 4.2. W.l.o.g. assume that $n \geq 2$. By proposition 4.1, it suffices to show that

(4.21) $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or

(4.22.i) $\exists \underset{\sim}{e}$ ($\underset{\sim}{e} \leq \underset{\sim}{a}_i$ and $\underset{\sim}{e} \in ENC^{\underset{\sim}{b}_i}$)

for some i, $1 \leq i \leq n$, holds.

Choose r.e. sets $A_i \in \underset{\sim}{a}_i$ and $B_i \in \underset{\sim}{b}_i$ such that $A_i, B_i \subseteq \omega^{(i)}$ and fix effective enumerations $\{A_{i,s} : s < \omega\}$ and $\{B_{i,s} : s < \omega\}$.

of A_i and B_i, respectively ($1 \leq i \leq n$). Set $A = \bigcup_{1 \leq i \leq n} A_i$ and $B = \bigcup_{1 \leq i \leq n} B_i$. Obviously, $dg(A) = a_0 \cup \ldots \cup a_n$, $dg(B) = b_0 \cup \ldots \cup b_n$, and $\{A_s : s < \omega\}$ and $\{B_s : s < \omega\}$ are effective enumerations of A and B, respectively, where $A_s = \bigcup_{1 \leq i \leq n} A_{i,s}$ and $B_s = \bigcup_{1 \leq i \leq n} B_{i,s}$.

Let $\bar{e}$ denote the (n+1)-tupel $(e_0,\ldots,e_n)$, where $e = \langle e_0,\ldots,e_n\rangle$, and let $\bar{e} \cap i$ be the i-tupel $(e_0,\ldots,e_{i-1})$ ($1 \leq i \leq n+1$).

As in the preceding proof, a set D witnesses (4.21) if D is Δ_2^0 and $D \not\leq_T A$. A witness set for (4.22.i), $1 \leq i \leq n$, is an r.e. set E such that $E \leq_T A_i$, $E^{(e)} \not\leq_T W_e$ uniformly in e for all $e < \omega$, and if $W_e \not\leq_T B_i$ then $E^{(e)} \not\leq_T B_i$ for all $e < \omega$. Obviously the existence of a witness set for (4.21) or (4.22.i) implies that the corresponding condition holds.

To show that (4.21) or (4.22.i) for some i, $1 \leq i \leq n$, is true, we will, as in the previous proof, simultaneously construct sets which are intended as witnesses for the individual conditions: a single set D for (4.21) and a sequence $\{E_{(e_0,\ldots,e_{i-1})} : e_0,\ldots,e_{i-1} < \omega\}$ of sets for (4.22.i), $1 \leq i \leq n$.

We will guarantee by the construction that D is Δ_2^0 and, for all i s.t. $1 \leq i \leq n$, and all $e_0,\ldots e_{i-1}$, the set $E_{(e_0,\ldots e_{i-1})}$ is r.e. and recursive in A_i.

Hence to have D as a witness for (4.21) it suffices to meet the requirements

$$\hat{R}_e : \quad D \neq \{e\}^A$$

for all $e < \omega$. To make $E_{(e_0,\ldots,e_{i-1})}$ witness (4.22.i) it suffices to guarantee that, for all $e < \omega$,

$$E^{(e)}_{(e_0,\ldots,e_{i-1})} \leq_T W_e \text{ uniformly in e,}$$

and to meet the requirements

$\hat{R}^{(e_0,\ldots,e_{i-1})}_{j}$: $W_{(j)_0} \not\leq_T B_i \Rightarrow E^{((j)_0)}_{(e_0,\ldots,e_{i-1})} \neq \{(j)_1\}^{B_i}$

for all $j<\omega$.

Hence at least one of the sets we will construct will meet its task as witness for one of the conditions if we meet the requirements

$\bar{R}_e$: $\hat{R}_{e_0}$ or $\exists i(\ 1\leq i\leq n$ and $\hat{R}^{\bar{e}\cap i}_{e_i}\ )$

(i.e. R_e : $D \neq \{e_0\}^A$ or $\exists i(\ 1\leq i\leq n$ and ($W_{(e_i)_0} \not\leq_T B_i \Rightarrow$

$\Rightarrow \quad E^{((e_i)_0)}_{\bar{e}\cap i} \neq \{(e_i)_1\}^{B_i}$)))

and satisfy the conditions

(4.23) If $D \neq \{e_0\}^A$ and

$\forall j\ (\ 1\leq j<i \Rightarrow (\ W_{(e_j)_0} \not\leq_T B_i$ and $E^{((e_j)_0)}_{(e_0,\ldots,e_{j-1})} =$

$= \{(e_j)_1\}^{B_j}\ ))$,

then, for all $k < \omega$, $E^{(k)}_{(e_0,\ldots,e_{i-1})} \leq_T W_k$ uniformly in k.

for all $e_0,\ldots e_n$ and $e < \omega$ s.t. $e = \langle e_0,\ldots,e_n\rangle$ and for all i s.t. $1\leq i\leq n$.

As in the previous proof fix a recursive function f which satisfies (4.15) and (4.16). Also the function $\lambda x,y([x]_y)$ is defined as there.

The construction depends on m, $m<\omega$, as in the proof of theorem 4.1. For all $e,s<\omega$, an attack on the requirement R_e will exist at stage s. This attack is in some state, where a state is a number less than or equal to n+1.(If one compares the case n = 2 of the following construction with the construction in the proof of theorem 4.1, the states have the following meaning expressed in the earlier terminology :

state n+1 = 3 - no attack

state 2 - attack which is not yet advanced

state 1 - advanced attack which is not yet completed

state 0 - completed attack.)

m-construction ($m < \omega$):

Stage s=0. For all $e < \omega$ let the attack on R_e be in state n+1 and let 0 be the A-, B- and B_i-information ($1 < i \leq n$) of the attack.

Stage s+1. For all $e = \langle e_0, \ldots, e_n \rangle \leq s+1$ let k be the state of the attack on R_e, let u, v, v_i be the A-, B-, and B_i-information, respectively, and let, if defined, x_i be the B_i-attacker, all at the end of stage s ($1 \leq i \leq n$).

Case 1 : $k = n+1$ and there exist numbers x and u' s.t.

$$\{e_0\}^{A_s}_s(e) = 0,$$

$$\mathbb{D}_{u'} = \{ z : z < u(e_0, A_s, e, s) \text{ and } z \notin A_s \}, \tag{4.24}$$

$$\begin{cases} x \geq u', \\ x = \langle (e_n)_0, e, x' \rangle \text{ for some } x' < \omega, \\ \{(e_n)_1\}^{B_n,s}_s(x) = 0, \text{ and} \\ \exists y < x \; (y \in W_{(e_n)_0,s} \setminus W_{(e_n)_0,s-1}). \end{cases} \tag{4.25}$$

Then fix u', x_n and v'_n such that u' satisfies (4.24), x_n is the least number x which satisfies (4.25) and v'_n is defined by

$$\mathbb{D}_{v'_n} = \{ z : z < u((e_n)_1, B_{n,s}, x_n, s),\ z \notin B_{n,s} \text{ and } z \in \omega^{(n)} \}.$$

Put the attack on R_e in state n, let x_n be the B_n-attacker of the attack, and replace the A- and B_n-information by u' and v'_n, respectively.

Case 2 : $1 < k \leq n$, $A_s \cap \mathbb{D}_u = \emptyset$ and there exist j and x such that

$$k \leq j \leq n,$$

$$\forall l \; (j < l \leq n \implies B_s \cap \mathbb{D}_{v_l} = \emptyset),$$

$$\begin{cases} x = \langle (e_j)_0, e, x' \rangle \text{ for some } x' < \omega, \\ x \geq u, \end{cases}$$

$$(4.26)\quad \begin{cases} \{(e_j)_1\}^{B_{s}}_{1} j,s\ (x) = 0, \text{ and} \\ \exists y < x\ (\ y \in W_{(e_j)_o,s} \setminus W_{(e_j)_o,s-1}\ ). \end{cases}$$

Then let j_o be the greatest such j and let x'_{j_o} be the least x which satisfies (4.26) for $j = j_o$. Define v'_{j_o} by

$$\mathbb{D}_{v'_{j_o}} = \{\, z : z < u((e_{j_o})_1, B_{j_o,s}, x'_{j_o}, s),\ z \notin B_{j_o,s} \text{ and } z \in \omega^{(j_o)}\}.$$

Put the attack on R_e in state j_o, replace the B_{j_o}-information and the B_{j_o}-attacker by v'_{j_o} and x'_{j_o}, respectively, and for all i s.t. $1 < i < j_o$, cancel the B_i-attacker of the attack and replace the B_i-information by 0.

Case 3 : Case 2 doesn't hold, $1 < k \leq n$, $A_s \cap \mathbb{D}_u = B_s \cap \mathbb{D}_{v_i} = \emptyset$ for all i st. $1 < i \leq n$, and there exists an x s.t. (4.26) holds for $j = k-1$.

Then let x_{k-1} be the least such x and let v'_{k-1} be defined by

$$\mathbb{D}_{v'_{k-1}} = \{\, z : z < u((e_{k-1})_1, B_{k-1,s}, x_{k-1}, s),\ z \notin B_{k-1,s} \text{ and } z \in \omega^{(k-1)}\}.$$

Subcase 3.1 : $k-1 > 1$. Put the attack on R_e in state $k-1$, let x_{k-1} be the B_{k-1}-attacker of the attack and replace the old B_{k-1}-information by v'_{k-1}.

Subcase 3.2 : $k-1 = 1$. Then fix v such that $\mathbb{D}_v = \bigcup_{1<i\leq n} \mathbb{D}_{v_i} \cup \mathbb{D}_{v'_1}$, put $\langle e,v\rangle$ into V and simultaneously enumerate $W_{f(\lceil m \rceil_e)}$ and B until either v shows up in $W_{f(\lceil m \rceil_e)}$ or an element of $\mathbb{D}_v$ shows up in B. In the first case put the attack in state 1, appoint x_1 as B_1-attacker and replace the B-information of the attack by v; further put e into D. In the second case do nothing.

Case 4 : $k = 1$, $B_s \cap \mathbb{D}_v = \emptyset$ and $(A_s \setminus A_{s-1}) \cap \mathbb{D}_u \neq \emptyset$. Then put the attack in state 0. If the least element of $(A_s \setminus A_{s-1}) \cap \mathbb{D}_u$ is in $\omega^{(i)}$, where $1 \leq i \leq n$, put the B_i-attacker of the attack into $E_{\overline{e}\cap i}$.

Case 5 : Either $k > 1$ and $A_s \cap \mathbb{D}_u \neq \emptyset$ or $k \leq 1$ and $B_s \cap \mathbb{D}_v \neq \emptyset$. Then put the attack in state n+1, cancel all attackers of the attack, and replace all informations of the attack by 0. If $k \leq 1$, extract e from D.

Case 6 : Case 2 doesn't hold, $k > 1$, $A_s \cap \mathbb{D}_u = \emptyset$ and there exists a j such that $k \leq j \leq n$ and $\mathbb{D}_{v_j} \cap B_s \neq \emptyset$. Then let j_o be the largest such j. Put the attack in state j_o+1 and, for all i s.t. $k \leq i \leq j_o$, cancel the B_i-attacker of the attack and replace the old B_i-information by 0.

Case 7 : Otherwise. Do nothing.

(State, attackers and informations of an attack at stage s+1 are, if not stated otherwise above, the same as at stage s.)

Since the m-construction is effective uniformly in m, $m < \omega$, as in the previous proof we can fix some m such that $V = W_m$. To show that for this m the m-construction yields the sets with the desired properties, it suffices to prove the following lemmata. Their proofs are similar to that of the analogous lemmata in the proof of theorem 4.1. (Lemma 4.15 is proved by induction on i. The inductive step resembles the proof of lemma 4.6.)

Lemma 4.10. Every stage of the construction is completed (in finitely many steps).

Lemma 4.11. For all $e > \omega$ and i s.t. $1 \leq i \leq n$, $E_{\bar{e}} \cap i$ is r.e. and $E_{\bar{e}} \cap i \leq_T A_i$.

Lemma 4.12. For all $e < \omega$, there are only finitely many stages at which the attack on H_e is put in state 1.

Lemma 4.13. D is a Δ^0_2 - set.

Lemma 4.14. Let $D = \{e_0\}^A$, $e = \langle e_0, \ldots, e_n \rangle$, and let s be a stage at which the attack on R_e is in state 1. Then there is a stage $t > s$ at which the attack on R_e is in state 0 or n+1.

Lemma 4.15. Let $e = \langle e_0, \ldots e_n \rangle$, $1 < i \leq n$, $D = \{e_0\}^A$ and for all j s.t. $1 \leq j < i$, let $\hat{R}_{e_j}^{\bar{e} \cap j}$ not be met. Further let s be a stage at which the attack on R_e is in state i. Then there is a stage $t > s$ at which the attack on R_e is in state 0 or in a state which is larger than i.

Lemma 4.16. If $e = \langle e_0, \ldots, e_n \rangle$ and $D = \{e_0\}^A$, then $E_{e \cap 1}^{(k)} \leq_T W_k$ uniformly in k for all $k < \omega$.

Lemma 4.17. If $e = \langle e_0, \ldots, e_n \rangle$, $1 < i \leq n$, $D = \{e_0\}^A$ and, for all j s.t. $1 \leq j < i$, $\hat{R}_{e_j}^{\bar{e} \cap j}$ is not met, then $E_{e \cap i}^{(k)} \leq_T W_k$ uniformly in k for all $k < \omega$.

Lemma 4.18. The requirement R_e is met. □

Remarks. By the splitting theorem it doesn't suffice to demand in theorem 4.1 that $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ are low (instead of $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ low). Even if we require in addition that, for $i \leq 1$, $\underset{\sim}{b}_i < \underset{\sim}{a}_i$, the conclusion of the theorem may not be true. This is a consequence of the following fact we proved:

(4.27) There exist low r.e. degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ such that both degrees are branching and $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1 = \underset{\sim}{0}'$.

(The proof of (4.27) resembles that of the theorems in §§ 6 and 7.) We do not know if the following extension of theorem 4.1 is true.

(4.28) If $\underset{\sim}{a}_0, \underset{\sim}{b}_0, \underset{\sim}{a}_1$ and $\underset{\sim}{b}_1$ are r.e. degrees s.t. $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ are low and $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1 < \underset{\sim}{0}'$, then either $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$ or $\underset{\sim}{a}_1 \in \mathrm{NC}^{\underset{\sim}{b}_i}$ for some $i \leq 1$.

§ 5 STRONGLY NONCAPPABLE DEGREES AND PAIRS OF R.E. DEGREES WITHOUT INFIMA.

In §4 we have seen that there exists an r.e. degree $\underset{\sim}{a} \neq \underset{\sim}{0},\underset{\sim}{0}'$ such that

$$\forall \underset{\sim}{b}\ (\ \underset{\sim}{b} < \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ low} \implies \underset{\sim}{a} \text{ is non-}\underset{\sim}{b}\text{-cappable}\ )$$

(cf. Corollary 4.4). Answering a question of Jockusch, we extend this result and show that there exists an r.e. degree $\underset{\sim}{a} \neq \underset{\sim}{0},\underset{\sim}{0}'$ s.t.

(5.1) $\forall \underset{\sim}{b} < \underset{\sim}{a}$ ($\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable)

(Theorem 5.1). Degrees $\underset{\sim}{a}$ which satisfy (5.1) are called strongly noncappable. The existence of a strongly noncappable degree $\neq \underset{\sim}{0},\underset{\sim}{0}'$ implies that not every r.e. degree $\neq \underset{\sim}{0},\underset{\sim}{0}'$ is half of a pair of incomparable r.e. degrees which have an infimum (Corollary 5.1). This extends Yates' result (in [33]) that not every r.e. degree $\neq \underset{\sim}{0},\underset{\sim}{0}'$ is half of a minimal pair.

By combining the construction of a strongly noncappable degree with the splitting technique (cf. Theorem 5.2), we get that to every r.e. degree $\neq \underset{\sim}{0},\underset{\sim}{0}'$ an incomparable strongly noncappable degree exists (Corollary 5.2). This implies a positive answer to another question of Jockusch (raised at the Conference on Mathematical Logic, Storrs, 1979), whether every r.e. degree $\neq \underset{\sim}{0},\underset{\sim}{0}'$ is half of a pair of r.e. degrees which have no infimum (cf. Corollary 5.3).

In [7] Lachlan proved that every nonzero r.e. degree can be split into two incomparable r.e. degrees which have an infimum. We prove the dual theorem : Every nonzero r.e. degree $\underset{\sim}{a}$ can be split into two r.e. degrees which have no infimum (Theorem 5.3). Indeed

this can be done above any low degree $\underset{\sim}{b} < \underset{\sim}{a}$ (Theorem 5.4). Some further results on strongly noncappable degrees are mentioned.

Definition 5.1. An r.e. degree $\underset{\sim}{a}$ is strongly noncappable if for all $\underset{\sim}{b} < \underset{\sim}{a}$, $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable.

Notice that $\underset{\sim}{0}$ and $\underset{\sim}{0}'$ are strongly noncappable, and that every strongly noncappable degree $\neq \underset{\sim}{0}$ is noncappable. Further if $\underset{\sim}{a}$ is strongly noncappable and $\underset{\sim}{a} \mid \underset{\sim}{b}$ then $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist.

Theorem 5.1. There exists a strongly noncappable degree $\underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}'$.

Corollary 5.1. Not every r.e. degree $\neq \underset{\sim}{0}, \underset{\sim}{0}'$ is half of a pair of incomparable r.e. degrees which have an infimum.

Proof. By the theorem fix a strongly noncappable degree $\underset{\sim}{a} \neq \underset{\sim}{0}, \underset{\sim}{0}'$. As remarked above, for all $\underset{\sim}{b}$ s.t. $\underset{\sim}{b} \mid \underset{\sim}{a}$, $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist.

Proof of theorem 5.1. Let B be any nonrecursive r.e. set and let $\{B_s : s < \omega\}$ be an effective enumeration of B. We will construct a nonrecursive r.e. set A such that $B \not\leq_T A$ and $\underset{\sim}{a} = dg(A)$ is strongly noncappable.

To make A nonrecursive, we insure that the first row $A^{(0)}$ of A is nonrecursive by meeting the requirements

P_e : $A^{(0)} \neq \{e\}$

for all $e < \omega$.

$B \not\leq_T A$ will follow if we meet the requirements

N_e : $B \neq \{e\}^A$

for all $e < \omega$.

The main task is to make $\underset{\sim}{a}$ strongly noncappable. We will construct r.e. sets E_e, $e < \omega$, s.t. for all $e < \omega$, $e = \langle e_0, e_1, e_2 \rangle$,

(5.2) $\quad (W_{e_0} = \{e_2\}^A$, $A \not\leq_T W_{e_0}$, and $W_{e_1} \not\leq_T A) \implies$

$$\implies \quad (E_e \not\leq_T W_{e_1}, A \text{ and } E_e \not\leq_T W_{e_0}).$$

This suffices to make $\underset{\sim}{a}$ strongly noncappable : Fix $\underset{\sim}{b}$ and $\underset{\sim}{c}$ such that $\underset{\sim}{b} < \underset{\sim}{a}$ and $\underset{\sim}{c} \not\leq \underset{\sim}{b}$, to show that there exists a $\underset{\sim}{d}$ s.t. $\underset{\sim}{d} \leq \underset{\sim}{c}, \underset{\sim}{a}$ and $\underset{\sim}{d} \not\leq \underset{\sim}{b}$. Choose e_0, e_1, e_2, and e s.t. $W_{e_0} \in \underset{\sim}{b}$, $W_{e_1} \in \underset{\sim}{c}$, $W_{e_0} = \{e_2\}^A$, and $e = \langle e_0, e_1, e_2 \rangle$. Since $\underset{\sim}{b} < \underset{\sim}{a}$, $A \not\leq_T W_{e_0}$. If $W_{e_1} \not\leq_T A$ then (5.2) implies that $E_e \not\leq_T W_{e_1}, A$ and $E_e \not\leq_T W_{e_0}$, i.e. $\underset{\sim}{d} = dg(E_e)$ will do. Otherwise $\underset{\sim}{d} = dg(W_{e_1}) = \underset{\sim}{c}$ will do.

To guarantee (5.2), we will meet the requirements

$$R_{\langle e,i \rangle} : \quad (W_{e_0} = \{e_2\}^A, A \not\leq_T W_{e_0}, \text{ and } W_{e_1} \not\leq_T A) \implies E_e \neq \{i\}^{W_{e_0}}$$

and satisfy the conditions

(5.3) $\quad E_e \not\leq_T W_{e_1}$,

and

(5.4) $\quad W_{e_0} = \{e_2\}^A \implies E_e \not\leq_T A$

for all $e_0, e_1, e_2, e, i < \omega$ s.t. $e = \langle e_0, e_1, e_2 \rangle$.

The priority ordering of the requirements is

$$N_0 > R_0 > P_0 > \cdots > N_{n+1} > R_{n+1} > P_{n+1} > \cdots \quad .$$

The negative requirements N_e are attacked using the Sacks preservation strategy. Recursive length- and restraint-functions l and r are defined by

$$l(e,s) = \max \{ x : \forall y < x \ (B_s(y) = \{e\}^{A_s}_s (y)) \} ,$$

$$r(e,s) = \max \{ u(e, A_s, x, s) : x \leq l(e,s) \} .$$

By the standard argument, N_e will be met if at only finitely many stages $s+1$ numbers less than $r(e,s)$ enter A.

To meet the requirements P_e we try to find a witness $\langle 0,e,y\rangle$ for P_e, i.e. a number y such that $A(\langle 0,e,y\rangle) \neq \{e\}(\langle 0,e,y\rangle)$.

The requirement P_e requires attention at stage s+1 if

(5.5) $\forall y \quad (\{e\}_s(\langle 0,e,y\rangle) = 0 \Rightarrow A_s(\langle 0,e,y\rangle) = 0)$

and there exists some $x < \omega$ such that

(5.6) $\{e\}_s(\langle 0,e,x\rangle) = 0$ and

$$\langle 0,e,x\rangle > \max\{\max\{r(f,s),R(f,s)\} : f \leq e\},$$

where R(f,s) is a restraint function in favor of the requirement R_f, which is defined during the construction.
If P_e requires attention at stage s+1 and x satisfies (5.6) then P_e requires attention through x at stage s+1.

To attack the requirements $R_{\langle e,i\rangle}$, we look for numbers $\langle i,x\rangle$, $x<\omega$, and stages s such that $\{i\}_s^{W_{e_0},s}(\langle i,x\rangle) = 0$. Then we want to put $\langle i,x\rangle$ into E_e and preserve the computation $\{i\}_s^{W_{e_0},s}(\langle i,x\rangle)$. If $W_{e_0} = \{e_2\}^A$ the latter can be done by putting a restraint on A. The necessity to make E_e recursive in A and W_{e_1} limits the numbers $\langle i,x\rangle$ which are allowed to enter E_e at a stage s.
$E_e \leq_T W_{e_1}$ is achieved by permitting. To get $E_e \leq_T A$, with every number enumerated in E_e simultaneously a "trace" is enumerated in A.

$R_{\langle e,i\rangle}$, $e = \langle e_0,e_1,e_2\rangle$, requires attention at stage s+1 if there exist numbers x,t,u and v such that the following conditions hold :

(5.7) $t \leq s$ and $\forall t' (t \leq t' \leq s \Rightarrow R(\langle e,i\rangle,t') = 0)$

(5.8) $\{i\}_t^{W_{e_0},t}(\langle i,x\rangle) = 0$

(5.9) $\exists y < \langle i,x\rangle (y \in A_t \setminus A_{t-1})$

(5.10) $u = u(i,W_{e_0,t},\langle i,x\rangle,t)$ and $W_{e_0,t}[u] = W_{e_0,s}[u]$

(5.11) $W_{e_0,s}[u] = \{e_2\}_s^{A_s}[u]$ and $v = \max\{u(e_2,A_s,y,s) : y < u\}$

(5.12) $\exists z < \langle i,x\rangle (z \in W_{e_1,s} \setminus W_{e_1,s-1})$

(5.13) $\langle e+1,i,x,t,v\rangle \geq \max\{\max\{r(f,s),R(f,s)\} : f \leq \langle e,i\rangle\}$.

If $R_{\langle e,i\rangle}$ requires attention at stage s+1 and if x,t,u and v satisfy (5.7) - (5.13) then $R_{\langle e,i\rangle}$ requires attention through $\langle x,t,u,v\rangle$ at stage s+1.

We can now describe the construction :

Stage 0. For all $x \langle \omega$, $R(x,0) = 0$.

Stage s+1. If no requirement requires attention at stage s+1 set $R(x,s+1) = R(x,s)$ for all $x \langle \omega$. Otherwise choose the requirement of highest priority which requires attention at stage s+1.

Case 1 : P_e is this requirement. Then let x be the least number through which P_e requires attention at stage s+1. Put $\langle 0,e,x\rangle$ into A and set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f \leq e \\ 0 & \text{otherwise.} \end{cases}$$

We say P_e receives attention (through x) at stage s+1.

Case 2 : $R_{\langle e,i\rangle}$ is this requirement and $e = \langle e_0,e_1,e_2\rangle$. Then let $\langle x,t,u,v\rangle$ be the least number through which $R_{\langle e,i\rangle}$ requires attention. Put $\langle i,x\rangle$ into E_e and $\langle e+1,i,x,t,v\rangle$ into A. Set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f < e \\ v & \text{if } f = e \\ 0 & \text{if } f > e. \end{cases}$$

We say $R_{\langle e,i\rangle}$ receives attention through $\langle x,t,u,v\rangle$ at stage s+1.

The construction is effective and thus the sets A and E_e, $e \langle \omega$, are recursively enumerable. That they have the desired properties will be shown in the following lemmata.

Lemma 5.1. Every requirement requires attention at most finitely often.

Proof by induction on the priority rank. For the inductive step

fix a requirement R. By induction hypothesis choose a stage s_0 after which no higher priority requirement requires attention.

Case 1 : $R = P_e$. If P_e requires attention at a stage $s_1 > s_0$ then it receives attention and (5.5) doesn't hold for any $s \geq s_1$. Hence P_e never requires attention after stage s_1.

Case 2 : $R = R_{\langle e,i\rangle}$. If $R_{\langle e,i\rangle}$ requires attention at a stage $s_1 > s_0$ then it receives attention and thus $R(\langle e,i\rangle,s_1) > 0$. Then, by choice of s_0, $R(\langle e,i\rangle,s) > 0$ for all $s \geq s_1$, i.e. (5.7) doesn't hold for any $s \geq s_1$. Hence $R_{\langle e,i\rangle}$ never requires attention after stage s_1.

Lemma 5.2. For all $e < \omega$:

(a) N_e is met

(b) $\lim_s r(e,s) < \omega$ exists

(c) $\lim_s R(e,s) < \omega$ exists.

Proof. Since $R(e,s+1) \neq R(e,s)$ only if R_e or a higher priority requirement receives attention, (c) follows from lemma 5.1. By the same lemma, only at finitely many stages s numbers less than $r(e,s)$ are put into A. Hence (a) and (b) can be proved by the standard arguments (cf. e.g. the proof of lemma 2.2).

Let $r(e) = \lim_s r(e,s)$ and $R(e) = \lim_s R(e,s)$.

Lemma 5.3. For all $e<\omega$, P_e is met.

Proof. Fix e and assume for a contradiction that P_e is not met, i.e. $A^{(e)} = \{e\}$.

By construction,

$$\forall x \; (A(\langle 0,e,x\rangle) = 1 \implies \{e\}(\langle 0,e,x\rangle) = 0).$$

Hence by assumption,

$$(5.15) \quad \forall x \; (A(\langle 0,e,x\rangle) = \{e\}(\langle 0,e,x\rangle) = 0).$$

It follows that (5.5) holds for all $s<\omega$. By lemmata 5.1 and 5.3

fix a stage s_0 s.t. P_e doesn't require attention after s_0 and

$\forall s \geq s_0 \forall f \leq e$ ($r(f,s) = r(f)$ and $R(f,s) = R(f)$).

Let $x_0 = \max \{ \max \{ r(f), R(f) \} : f \leq e \}$. By (5.15) choose $x \geq x_0$ and $s \geq s_0$ s.t. $\{e\}_s(\langle 0,e,x \rangle) = 0$. Then P_e requires attention through x at stage s+1 contrary to the choice of s_0.

Lemma 5.4. For all $e,i < \omega$, $R_{\langle e,i \rangle}$ is met.

Proof. Fix $e,i < \omega$, $e = \langle e_0, e_1, e_2 \rangle$. We distinguish the following two cases.

Case 1 : $R(\langle e,i \rangle) > 0$. Let s be the least stage s.t. for all $s' > s$, $R(\langle e,i \rangle, s') > 0$. Then there are x,t,u,v such that $R_{\langle e,i \rangle}$ receives attention through $\langle x,t,u,v \rangle$ at stage s+1. Hence $\langle i,x \rangle \in E_{e,s+1}$, $\{i\}_s^{W_{e_0,s}}(\langle i,x \rangle) = 0$, $u = u(i, W_{e_0,s}, \langle i,x \rangle, s)$, $W_{e_0,s}[u] = \{e_2\}_s^{A_s}[u]$, $v = \max \{ u(e_2, A_s, y, s) : y < u \}$, and $R(\langle e,i \rangle, s+1) = v$. Since $R(\langle e,i \rangle, s') > 0$ for all $s' > s$, neither $R_{\langle e,i \rangle}$ nor a higher priority requirement receives attention after stage s+1. Hence $R(\langle e,i \rangle, s') = v$, for all $s' > s$, and $A_s[v] = A[v]$ (Note that at stage s+1 only $\langle e+1,i,x,t,v \rangle$ enters A.). The latter implies $\{e_2\}_s^{A_s}[u] = \{e_2\}^A[u]$. If $W_{e_0} \neq \{e_2\}^A$ then $R_{\langle e,i \rangle}$ is met trivially. Otherwise, by the above,

$$W_{e_0,s}[u] = \{e_2\}_s^{A_s}[u] = \{e_2\}^A[u] = W_{e_0}[u]$$

and thus

$$\{i\}^{W_{e_0}}(\langle i,x \rangle) = \{i\}_s^{W_{e_0,s}}(\langle i,x \rangle) = 0.$$

It follows that $E_e \neq \{i\}^{W_{e_0}}$, i.e. $R_{\langle e,i \rangle}$ is met.

Case 2 : $R(\langle e,i \rangle) = 0$. Assume for a contradiction that $R_{\langle e,i \rangle}$ is not met, i.e. $W_{e_0} = \{e_2\}^A$, $A \not\leq_T W_{e_0}$, $W_{e_1} \not\leq_T A$, and $E_e = \{i\}^{W_{e_0}}$. By lemmata 5.1 and 5.2 and by the assumptions $R(\langle e,i \rangle) = 0$ and $E_e = \{i\}^{W_{e_0}}$, fix numbers s' and x' such that

(5.16) $R_{\langle e,i \rangle}$ doesn't require attention after stage s',

(5.17) $\forall s \geq s' (R(\langle e,i \rangle, s) = 0)$,

(5.18) $x' \geq \max\{\max\{r(f,s), R(f,s)\} : f \leq \langle e,i\rangle, s < \omega\}$, and

(5.19) $\forall x \geq x'(\{i\}_s^{W_{e_o}}(\langle i,x\rangle) = 0)$.

By (5.19) there are W_{e_o} - recursive functions $\lambda x(s(x))$ and $\lambda x(u(x))$ s.t. , for all $x \geq x'$, $s(x) \geq s'$ and

(5.20) $\forall s \geq s(x)\ (\{i\}_s^{W_{e_o},s}(\langle i,x\rangle) = 0$,

$u(x) = u(i, W_{e_o,s}, i, x, s)$ and $W_{e_o,s}[u(x)] = W_{e_o}[u(x)])$.

Since $A \not\equiv_T W_{e_o}$, there exist infinitely many $x < \omega$ s.t. $A_{s(x)}[\langle i,x\rangle] \neq A[\langle i,x\rangle]$. Therefore there are sets $\{x_n : n < \omega\}$ and $\{t_n : n < \omega\}$ such that, for all $n < \omega$, $x' \leq x_n < x_{n+1}$, $t_n \geq s(x_n)$ and

(5.21) $\exists y < \langle i,x_n\rangle(y \in A_{t_n} \setminus A_{t_n-1})$.

Since $W_{e_o} = \{e_2\}^A$, i.e. $W_{e_o} \leq_T A$, the functions $\lambda x(s(x))$ and $\lambda x(u(x))$ are recursive in A and hence the sets $\{x_n : n < \omega\}$ and $\{t_n : n < \omega\}$ can be choosen to be r.e. in A. Also since $W_{e_o} = \{e_2\}^A$, A - r.e. sets $\{v_n : n < \omega\}$ and $\{s_n : n < \omega\}$ exist such that , for all $n < \omega$,

(5.22) $t_n \leq s_n$, $W_{e_o,s_n}[u(x_n)] = \{e_2\}_{s_n}^{A_{s_n}}[u(x_n)]$,

$v_n = \max\{u(e_2, A_{s_n}, y, s_n) : y < u(x_n)\}$, and

$A_{s_n}[v_n] = A[v_n]$.

Since $W_{e_1} \not\equiv_T A$, for infinitely many $n < \omega$, $W_{e_1,s_n}[\langle i,x_n\rangle] \neq W_{e_1}[\langle i,x\rangle]$. Fix such an n and a stage $s \geq s_n$ such that

(5.23) $\exists y < \langle i,x_n\rangle(y \in W_{e_1,s} \setminus W_{e_1,s-1})$.

Then, by (5.17)-(5.23), $R_{\langle e,i\rangle}$ requires attention through $\langle x_n, t_n, u(x_n), v_n\rangle$ at stage s+1. Since $s+1 > s'$, this is a contradiction to (5.16).

Lemma 5.5. For all $e < \omega$, $e = \langle e_o, e_1, e_2\rangle$:

(a) $E_e \leq_T W_{e_1}$

(b) $W_{e_o} = \{e_2\}^A \longrightarrow E_e \leq_T A$.

Proof. (a) : By (5.12),

$$\forall x \, (\, x \in E_{e,s+1} \setminus E_{e,s} \Rightarrow \exists z < x \, (\, z \in W_{e_1,s} \setminus W_{e_1,s-1} \,) .$$

Hence $E_e \leq_T W_{e_1}$ by permitting.

(b) : First notice that if $R_{\langle e,i \rangle}$ receives attention through $\langle x,t,u,v \rangle$ at stage s+1 then $\langle e+1,i,x,t,v \rangle \in A_{s+1} \setminus A_s$. That $\langle e+1,i,x,t,v \rangle \in A_{s+1}$, follows from the construction immediately. To see $\langle e+1,i,x,t,v \rangle \notin A_s$, fix s' such that $\langle e+1,i,x,t,v \rangle$ is put into A at stage s'+1. Then $R_{\langle e,i \rangle}$ receives attention through $\langle x,t,u',v \rangle$, for some $u' < \omega$, at stage s'+1. Hence $R(\langle e,i \rangle, s'+1) > 0$ and $t \leq s'$ (by (5.7)). Since $R_{\langle e,i \rangle}$ requires attention through $\langle x,t,u,v \rangle$ at stage s+1, by (the second part of) (5.7) follows that $s \leq s'$, and thus $\langle e+1,i,x,t,v \rangle \notin A_s$.

It follows for all $i,x,s < \omega$ that

(5.24) $\langle i,x \rangle \in E_{e,s+1} \setminus E_{e,s} \Rightarrow \exists t,u,v \, (\, R_{\langle e,i \rangle}$ requires attention through $\langle x,t,u,v \rangle$ at stage s+1 and $\langle e+1,i,x,t,v \rangle \in A_{s+1} \setminus A_s \,)$.

By (5.9) - (5.11), if $R_{\langle e,i \rangle}$ requires attention through $\langle x,t,u,v \rangle$ at stage s+1 then $t \leq t(\langle i,x \rangle)$, $u \leq u(\langle i,x \rangle)$ and $v \leq v(\langle i,x \rangle)$, where

$$t(\langle i,x \rangle) = \mu s \, (\, A_s \, [\langle i,x \rangle] = A[\langle i,x \rangle] \,),$$

$$u(\langle i,x \rangle) = \max \{ u(i, W_{e_0,s}, \langle i,x \rangle, s) : s \leq t(\langle i,x \rangle) \} , \text{ and}$$

$$v(\langle i,x \rangle) = \max \{ u(e_2, A_s, y, s) : y < u(\langle i,x \rangle), s < \omega \} .$$

By the assumption that $W_{e_0} = \{e_2\}^A$, $v(\langle i,x \rangle) < \omega$ is welldefined, and $\lambda y(t(y))$, $\lambda y(u(y))$ and $\lambda y(v(y))$ are total A-recursive functions.

By the above, (5.24) may be replaced by

(5.25) $\langle i,x \rangle \in E_{e,s+1} \setminus E_{e,s} \Rightarrow$

$$\Rightarrow \exists y \leq \langle e+1,i,x,t(\langle i,x \rangle), v(\langle i,x \rangle) \rangle \, (\, y \in A_{s+1} \setminus A_s) .$$

That E_e is recursive in A can now be seen as follows. To compute $E_e(p)$ recursively in A, for given $p < \omega$, recursively in A find a stage s such that

$$A_s[\langle e+1,(p)_0,(p)_1,t(p),v(p)\rangle] = A[\langle e+1,(p)_0,(p)_1,t(p),v(p)\rangle] .$$

Then, by (5.25), $E_e(p) = E_{e,s}(p)$.

Lemmata 5.2 (a) and 5.3 imply that $\underset{\sim}{a} \neq \underset{\sim}{0},\underset{\sim}{0}'$. In the first part of the proof we have seen that $\underset{\sim}{a}$ strongly noncappable follows from lemmata 5.4 and 5.5.

Note that the proof of Theorem 5.1 shows that for any nonzero r.e. degree $\underset{\sim}{b}$ a strongly noncappable degree $\underset{\sim}{a}$ exists which avoids the cone of $\underset{\sim}{b}$ (i.e. $\underset{\sim}{b} \not\leq \underset{\sim}{a}$). That to every such $\underset{\sim}{b} < \underset{\sim}{0}'$ an incomparable strongly noncappable degree exists, will be a consequence of the following splitting theorem.

Theorem 5.2. Let $\underset{\sim}{b}$ be a nonzero r.e. degree. Then there are strongly noncappable degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that

(i) $\underset{\sim}{0}' = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$, and

(ii) $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$.

Corollary 5.2. If $\underset{\sim}{b} \neq \underset{\sim}{0},\underset{\sim}{0}'$ then there exists a strongly noncappable degree $\underset{\sim}{a}$ such that $\underset{\sim}{a} \mid \underset{\sim}{b}$.

Proof. By the theorem let $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ be strongly noncappable degrees which satisfy (i) and (ii) for a fixed $\underset{\sim}{b} \neq \underset{\sim}{0},\underset{\sim}{0}'$. If $\underset{\sim}{a}_0 \mid \underset{\sim}{b}$, $\underset{\sim}{a} = \underset{\sim}{a}_0$ will do. Otherwise, by (ii), $\underset{\sim}{a}_0 < \underset{\sim}{b}$. By $\underset{\sim}{b} \neq \underset{\sim}{0}'$ and (i) follows $\underset{\sim}{a}_1 \not\leq \underset{\sim}{b}$, and thus, by (ii), $\underset{\sim}{a}_1 \mid \underset{\sim}{b}$. I.e. in this case $\underset{\sim}{a} = \underset{\sim}{a}_1$ will do.

Corollary 5.3. Every r.e. degree $\neq \underset{\sim}{0}, \underset{\sim}{0}'$ is half of a pair of r.e. degrees which have no infimum.

Proof. Fix $\underset{\sim}{b} \neq \underset{\sim}{0}, \underset{\sim}{0}'$. By corollary 5.2 choose a strongly noncappable degree $\underset{\sim}{a}$ s.t. $\underset{\sim}{b} \mid \underset{\sim}{a}$. Then $\underset{\sim}{b} \cap \underset{\sim}{a}$ doesn't exist.

Proof of theorem 5.2. The proof combines that of theorem 5.1 with that of the splitting theorem.

Fix $\underset{\sim}{b} > \underset{\sim}{0}$, an r.e. set $B \in \underset{\sim}{b}$, and an effective enumeration $\{B_s : s < \omega\}$ of B. Further fix a complete r.e. set $A \subseteq \omega^{(o)}$, and let f be a recursive function which enumerates A without repetitions. We will construct r.e. sets A_0 and A_1 whose degrees $\underset{\sim}{a}_0 = dg(A_0)$ and $\underset{\sim}{a}_1 = dg(A_1)$ will satisfy the theorem.

To guarantee (i), it suffices to show that $A \not\leq_T A_0 \oplus A_1$. For this purpose we insure that $A = A_0^{(o)} \cup A_1^{(o)}$. This is done as in the proof of the splitting theorem by putting $f(s)$ into $A_0^{(o)}$ or $A_1^{(o)}$ at stage $s+1$.

Condition (ii) is satisfied by the Sacks preservation strategy (as $B \not\leq_T A$ in the proof of theorem 5.1) : For all $e < \omega$, $j \leq 1$, we meet the requirements

N_{2e+j} : $B \neq \{e\}^{A_j}$.

$l(2e+j,s)$ and $r(2e+j,s)$ is defined as $l(e,s)$ and $r(e,s)$, respectively, in the previous proof; only A has to be replaced by A_j ($j \leq 1$, $e,s < \omega$). Again, if at only finitely many stages $s+1$ numbers less than $r(2e+j,s)$ are enumerated in A_j then N_{2e+j} is met.

$\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ are made strongly noncappable by the strategy of the preceding proof. We construct r.e. sets E_e^j, meet the requirements

$R_{2\langle e,i\rangle+j}$: ($W_{e_o} = \{e_2\}^{A_j}$, $A_j \not\leq_T W_{e_o}$, and $W_{e_1} \not\leq_T A_j$) $\Longrightarrow$

$\Longrightarrow$ $E_e^j \neq \{i\}^{W_{e_o}}$,

and satisfy the conditions

(5.26) $E_s^j \leq_T W_{e_1}$

and

(5.27) $W_{e_o} = \{e_2\}_1^{A_j} \Longrightarrow E_e^j \leq_T A_j$

for all $e,e_o,e_1,e_2,i < \omega, j \leq 1$, s.t. $e = \langle e_o,e_1,e_2\rangle$.

One can see, as in the proof of theorem 5.1, that this suffices to make the degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ strongly noncappable.

The requirement $R_{2\langle e,i\rangle+j}$, where $e = \langle e_o,e_1,e_2\rangle$ and $j \leq 1$, requires attention at stage s+1 if there exist numbers x,t,u and v s.t.

(5.28) $t \leq s$ and $\forall t'(t \leq t' \leq s \Longrightarrow R(2\langle e,i\rangle+j,t') = 0)$,

(5.29) $\{i\}_t^{W_{e_o},t}(\langle i,x\rangle) = 0$,

(5.30) $\exists y < \langle i,x\rangle (y \in A_{j,t} \setminus A_{j,t-1})$,

(5.31) $u = u(i,W_{e_o,t},\langle i,x\rangle,t)$ and $W_{e_o,t}[u] = W_{e_o,s}[u]$,

(5.32) $W_{e_o,s}[u] = \{e_2\}_s^{A_j,s}[u]$ and

$v = \max\{u(e_2,A_{j,s},y,s) : y < u\}$,

(5.33) $\exists z < \langle i,x\rangle (z \in W_{e_1,s} \setminus W_{e_1,s-1})$, and

(5.34) $\langle e+1,i,x,t,v\rangle \geq \max\{\max\{r(f,s),R(f,s)\} : f \leq 2\langle e,i\rangle+j\}$,

where the function R(f,s) is defined during the construction.

If $R_{2\langle e,i\rangle+j}$ requires attention at stage s+1 and x,t,u and v satisfy (5.28) - (5.34) then $R_{2\langle e,i\rangle+j}$ requires attention through $\langle x,t,u,v\rangle$ at stage s+1.

Construction.

Stage s=0. For all $y < \omega$, $R(y,0) = 0$.

Stage s+1. Let y' be the least number y such that $f(s) < \max\{r(y,s),R(y,s)\}$ and let z' be the least index of a re-

quirement which requires attention at stage s+1. If such a number y' (x') doesn't exist, set $y' = \omega$ ($x' = \omega$).

Case 1 : $z' \leq y', z' < \omega$. Let $z' = 2\langle e,i\rangle + j$, $e,i < \omega$, $j \leq 1$, and let $\langle x,t,u,v\rangle$ be the least number through which $R_{z'}$ requires attention at stage s+1. Put $\langle i,x\rangle$ into E_e^j, $\langle e+1,i,x,t,v\rangle$ into A_j, and $f(s)$ into A_{1-j}. Further set

$$R(y,s+1) = \begin{cases} R(y,s) & \text{if } y < z' \\ v & \text{if } y = z' \\ 0 & \text{if } y > z'. \end{cases}$$

We say that $R_{z'}$ receives attention through $\langle x,t,u,v\rangle$ at stage s+1.

Case 2 : $y' < z'$. Let $y' = 2e+j$, $e < \omega$, $j \leq 1$. Put $f(s)$ into A_{1-j} and set

$$R(y,s+1) = \begin{cases} R(y,s) & \text{if } y < y' \\ 0 & \text{otherwise.} \end{cases}$$

Case 3 : $y' = z' = \omega$. Put $f(s)$ into A_0 and set $R(y,s+1) = R(y,s)$ for all $y < \omega$.

Note that the construction is effective and thus A_j and E_e^j, $e < \omega$, $j \leq 1$, are recursively enumerable. To show the correctness of the construction we prove the following lemmata.

Lemma 5.6. $\underset{\sim}{0}' = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$.

Proof. By construction, $A = A_0^{(0)} \cup A_1^{(0)}$. This implies $A \leq_1 A_0 \oplus A_1$, and thus $\underset{\sim}{0}' = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$.

Lemma 5.7. For all $f,e < \omega$ and $j \leq 1$ s.t. $f = 2e+j$:

(a) The set $I_f = \{s : \exists x (x \in A_{j,s+1} \setminus A_{j,s}$ and $x < r(f,s))\}$ is finite.

(b) N_f is met.

(c) $\lim_s r(f,s) < \omega$ exists.

(d) R_f requires attention at most finitely often.

(e) $\lim_s R(f,s) < \omega$ exists.

Proof by simultaneous induction on f. For the inductive step fix f. (a), (b) and (c) are proved by the standard argument as used e.g. in the proof of lemma 2.2. To see (d) and (e), by induction hypothesis and (c), fix a stage s_0 such that no requirement $R_{f'}$, $f' < f$, requires attention after stage s_0,

$\forall s \geq s_0 \forall f' \leq f \; (\; f(s) \geq r(f',s) \;)$, and

$\forall s \geq s_0 \forall f' < f \; (\; f(s) \geq R(f',s) \;)$.

If R_f never requires attention after stage s_0 then (d) is immediate and $\lim_s R(f,s) = R(f,s_0)$. Otherwise let s_1 be the least stage after s_0 at which R_f requires attention. R_f receives attention at stage s_1, and by choice of s_0, $\forall s \geq s_1 \; (\; R(f,s) = R(f,s_1) > 0 \;)$. I.e. $\lim_s R(f,s) = R(f,s_1)$ and, by (5.28), R_f doesn't require attention after stage s_1.

Lemma 5.7 (b) implies that condition (ii) of the theorem is satisfied by the constructed degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$. It remains to show that $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ are strongly noncappable. As in the proof of theorem 5.1 this will follow from the two subsequent lemmata. They are proved as lemmata 5.4 and 5.5, respectively.

Lemma 5.8. For all $f < \omega$, R_f is met.

Lemma 5.9. For all $e < \omega$, $j \leq 1$, $e = \langle e_0, e_1, e_2 \rangle$:

(a) $E_e^j \leq_T W_{e_1}$

(b) $W_{e_0} = \{e_2\}^{A_j} \Rightarrow E_e^j \leq_T A_j$.

Remarks. As in the proof of theorem 2.2 one can see that the above constructed sets A_0 and A_1 are low. Hence we may assume that the degrees $\underset{\sim}{a}_0, \underset{\sim}{a}_1$ and $\underset{\sim}{a}$ in theorem 5.2 and comllary 5.2, respectively, are low.

The proof of theorem 5.2 can be combined with the techniques of § 4 to show the following extension of theorem 5.2 :

(5.35) Let $\underset{\sim}{a}_i, \underset{\sim}{b}_i$ and $\underset{\sim}{c}_i$, $i \leq n$ ($n < \omega$), be r.e. degrees such that $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n = \underset{\sim}{0}'$, $\underset{\sim}{b}_0 \cup \ldots \cup \underset{\sim}{b}_n$ is low, and, for all $i \leq n$, $\underset{\sim}{c}_i \not\leq \underset{\sim}{b}_i$. Then there exists an $i \leq n$ and r.e. degrees $\underset{\sim}{a}_{i0}$ and $\underset{\sim}{a}_{i1}$ such that

(i) $\underset{\sim}{a}_{i0}$ and $\underset{\sim}{a}_{i1}$ are strongly noncappable,

(ii) $\underset{\sim}{a}_{i0}$ and $\underset{\sim}{a}_{i1}$ are low,

(iii) $\underset{\sim}{a}_{i0} \cup \underset{\sim}{a}_{i1} = \underset{\sim}{a}_i \cup \underset{\sim}{b}_i$,

(iv) $\underset{\sim}{b}_i \leq \underset{\sim}{a}_{i0}, \underset{\sim}{a}_{i1}$, and

(v) $\underset{\sim}{c}_i \not\leq \underset{\sim}{a}_{i0}, \underset{\sim}{a}_{i1}$.

A consequence of (5.35) and the Robinson splitting theorem ([14], Theorem 3) is

(5.36) If $\underset{\sim}{b}$ is low and $\underset{\sim}{c} \not\leq \underset{\sim}{b}$ then there is a strongly noncappable (low) degree $\underset{\sim}{a}$ such that $\underset{\sim}{b} < \underset{\sim}{a}$ and $\underset{\sim}{c} \not\leq \underset{\sim}{a}$.

As in the proof of corollary 2.6 one can see that (5.36) implies that the set of the strongly noncappable degrees is an automorphism basis in $\underset{\sim}{R}$.

(5.35) and Shoenfield/Soare [19] (or Lachlan [7]) imply

(5.37) There exist strongly noncappable degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ s.t. $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ is not strongly noncappable.

(5.37) says that the set of the strongly noncappable degrees is not closed under finite suprema, and thus it isn't closed upwards in $\underset{\sim}{R}$. That it isn't closed downwards follows from the fact that every strongly noncappable degree $\neq \underset{\sim}{0}$ is noncappable by (1.9). The strongly noncappable degrees are closed under finite infima, however, since any pair of incomparable r.e. degrees of which one is strongly noncappable has no infimum.

An interesting question is, if Shoenfield's conjecture (cf. §1) restricted to the set of strongly noncappable degrees is true. Since for incomparable strongly noncappable degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist, it is probable that the counterexample given in § 1 for the set of the noncappable degrees doesn't work for the strongly noncappable degrees.

Theorem 5.3. Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be nonzero r.e. degrees. Then there are r.e. degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$, $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist, and $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$.

Proof. The proof is similar to the proof of theorem 5.2. It combines the proof of the splitting theorem with a new construction of a pair of r.e. degrees without infimum, which is based on the strongly noncappable degree construction. It seems that the earlier constructions of pairs without infimum by Jockusch, Lachlan (in [4]) and Yates (in [33]) cannot be combined with the splitting technique to prove theorem 5.3.

Since $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ implies that $\underset{\sim}{a}_0, \underset{\sim}{a}_1 \leq \underset{\sim}{a}$, w.l.o.g. we can assume that $\underset{\sim}{b} \leq \underset{\sim}{a}$. Choose r.e. sets A and B such that $A \in \underset{\sim}{a}$, $B \in \underset{\sim}{b}$, and $A \subseteq \omega^{(o)}$. Let f be an one-one recursive function which enumerates A and let $\{B_s : s < \omega\}$ be an effective enumeration of B. We will construct r.e. sets A_0 and A_1 such that $\underset{\sim}{a}_0 = dg(A_0)$ and $\underset{\sim}{a}_1 = dg(A_1)$ have the desired properties.

As in the preceding proof we will have $A = A_0^{(o)} \cup A_1^{(o)}$, and thus $\underset{\sim}{a} \leq \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$. That $\underset{\sim}{a}_0, \underset{\sim}{a}_1 \leq \underset{\sim}{a}$, and therefore $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 \leq a$, is guaranteed by permitting; i.e. at any stage s+1 only numbers $\geqslant f(s)$ are enumerated in A_0 and A_1.

To have $B \not\leq_T A_0, A_1$, i.e. $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$, we meet the requirements N_{2e+j} of the proof of theorem 5.2 for all $e < \omega$ and $j \leq 1$. The recursive functions l and r are defined as there.

Notice that by the above we get $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$, which implies that $\underset{\sim}{a}_0 \mid \underset{\sim}{a}_1$ (since $\underset{\sim}{b} \leq \underset{\sim}{a}$). Hence, to show that $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist, it suffices to satisfy for all r.e. degrees $\underset{\sim}{c}$

(5.38) $\quad \underset{\sim}{c} < \underset{\sim}{a}_1 \implies \exists \underset{\sim}{d}$ ($\underset{\sim}{d} \leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$ and $\underset{\sim}{d} \not\leq \underset{\sim}{c}$).

To satisfy (5.38), we construct r.e. sets E_e, meet the requirements

$R_{\langle e,i \rangle}$: $(W_{e_o} = \{e_1\}^{A_1}$ and $A_1 \not\leq_T W_{e_o}) \implies E_e \neq \{i\}^{W_{e_o}}$,

and satisfy the conditions

(5.39) $\quad E_e \leq_T A_0$, and

(5.40) $\quad W_{e_o} = \{e_1\}^{A_1} \implies E_e \leq_T A_1$,

for all $e, e_o, e_1, i < \omega$ s.t. $e = \langle e_o, e_1 \rangle$.

This suffices to guarantee (5.38): Given $\underset{\sim}{c} < \underset{\sim}{a}_1$, choose $e = \langle e_o, e_1 \rangle$ such that $W_{e_o} \in \underset{\sim}{c}$ and $W_{e_o} = \{e_1\}^{A_1}$. By the requirements $R_{\langle e,i \rangle}$, $i < \omega$, $E_e \not\leq_T W_{e_o}$ and, by (5.39) and (5.40) $E_e \leq_T A_0, A_1$. I.e. the degree $\underset{\sim}{d} = dg(E_e)$ witnesses (5.38).

The requirement $R_{\langle e,i \rangle}$ <u>requires attention</u> at stage s+1 if there are numbers x,t,u and v such that

(5.41) $\quad t \leq s$ and $\forall t' (t \leq t' \leq s \implies R(\langle e,i \rangle, t') = 0)$,

(5.42) $\quad \{i\}_t^{W_{e_o}}(\langle i,x \rangle) = 0$,

(5.43) $\quad u = u(i, W_{e_o,t}, i, x, t)$ and $W_{e_o,t}[u] = W_{e_o,s}[u]$,

(5.44) $\quad \exists y < \langle i,x \rangle (y \in A_{1,t} \setminus A_{1,t-1})$,

(5.45) $\quad W_{e_o,s}[u] = \{e_1\}_s^{A_{1,s}}[u]$ and

$$v = \max \{ u(e_1, A_{1,s}, y, s) : y < u \},$$

(5.46) $\quad f(s) < \langle i,x \rangle$, and

(5.47) $\quad \langle e+1, i, x, t, v \rangle \geq \max \{ \max \{ r(f,s), R(f,s) \} : f \leq \langle e,i \rangle \}$,

where the function R is defined during the construction.

If $R_{\langle e,i\rangle}$ requires attention at stage s+1 and x,t,u and v satisfy (5.41) - (5.47) then $R_{\langle e,i\rangle}$ requires attention through $\langle x,t,u,v\rangle$ at stage s+1.

Construction.

Stage s=0. For all $f < \omega$, $R(f,0) = 0$.

Stage s+1. Define y' and z' as in the construction of the preceding proof.

Case 1 : $z' \leq y', z' < \omega$. Let $z' = \langle e,i\rangle$ and let $\langle x,t,u,v\rangle$ be the least number through which $R_{z'}$ requires attention at stage s+1. Put $\langle i,x\rangle$ into E_e, $\langle e+1,i,x,t,v\rangle$ into A_1, and f(s) into A_0. Set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f < z' \\ v & \text{if } f = z' \\ 0 & \text{if } f > z'. \end{cases}$$

Case 2 : $y' < z'$. Let $y' = 2e+j$, $e < \omega$, $j \leq 1$. Put f(s) into A_{1-j} and set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f < y' \\ 0 & \text{otherwise.} \end{cases}$$

Case 3 : $y' = z' = \omega$. Put f(s) into A_0 and set $R(f,s+1) = R(f,s)$ for all $f < \omega$.

The construction is effective. Hence the sets A_0, A_1 and E_e are recursively enumerable. That they have the desired properties will follow from the subsequent lemmata.

Lemma 5.10. $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$.

Proof. By construction $A = A_0^{(o)} \cup A_1^{(o)}$, and thus $\underset{\sim}{a} \leq \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$. Further a number x enters A_0 at stage s+1 only if $x = f(s)$, and it enters A_1 only if $f(s) \leq x$. Hence $A_0, A_1 \leq_T A$ by permitting. It follows that $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$.

As in the preceding proof lemma 5.7 is true (by the old proof).

Lemma 5.11. For all $e,i < \omega$, $R_{\langle e,i\rangle}$ is met.

The proof of lemma 5.11 is essentially the same as that of lemma 5.4. In the case $R(\langle e,i\rangle) = 0$, under the assumption that $R_{\langle e,i\rangle}$ is not met, as in the proof of lemma 5.4 we can choose a stage s' after which $R_{\langle e,i\rangle}$ doesn't require attention, and an A_1-r.e. set $\{\langle x_n,t_n,u(x_n),v_n,s_n\rangle : n < \omega\}$ such that for all n

$x_n < x_{n+1}$, $s_n > s'$, and $R_{\langle e,i\rangle}$ requires attention through $\langle x_n,t_n,u(x_n),v_n\rangle$ at every stage $s+1 > s_n$ at which (5.46) holds for $x = x_n$, i.e. at which $f(s) < \langle i,x_n\rangle$.

Since by lemmata 5.10 and 5.7(b) $\underset{\sim}{a} \not\leq \underset{\sim}{a}_1$, there must be numbers n and s such that $s \geq s_n$ and $f(s) < \langle i,x_n\rangle$, i.e. $R_{\langle e,i\rangle}$ requires attention at stage $s+1$ contrary to the choice of s'.

Lemma 5.12. For all $e < \omega$, $e = \langle e_0,e_1\rangle$:

(a) $E_e \leq_T A_0$

(b) $W_{e_0} = \{e_1\}^{A_1} \Rightarrow E_e \leq_T A_1$.

Lemma 5.12 is proved as lemma 5.5. This completes the proof of theorem 5.3.

We conclude this section with an extension of Theorem 5.3.

Theorem 5.4. Let $\underset{\sim}{a},\underset{\sim}{b}$ and $\underset{\sim}{d}$ be r.e. degrees such that $\underset{\sim}{d} < \underset{\sim}{a}$, $\underset{\sim}{b} \not\leq \underset{\sim}{d}$, and $\underset{\sim}{d}$ is low. Then there exist r.e. degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ s.t. $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$, $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist, $\underset{\sim}{d} < \underset{\sim}{a}_0,\underset{\sim}{a}_1$, and $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0,\underset{\sim}{a}_1$.

An immediate consequence of Theorem 5.4 is the following partial density result for pairs without infimum.

Corollary 5.4. If $\underset{\sim}{a} < \underset{\sim}{b}$ and $\underset{\sim}{a}$ is low then there are r.e. degrees $\underset{\sim}{c}$ and $\underset{\sim}{d}$ such that $\underset{\sim}{a} < \underset{\sim}{c}, \underset{\sim}{d} < \underset{\sim}{b}$ and $\underset{\sim}{c} \cap \underset{\sim}{d}$ doesn't exist.

Proof of theorem 5.4. We modify the proof of theorem 5.4 in a similar way as we did the proof of theorem 2.1 to show theorem 2.2.

Choose r.e. sets $A \in \underset{\sim}{a}$, $B \in \underset{\sim}{b}$ and $D \in \underset{\sim}{d}$ s.t. $A \cap D = \emptyset$ and $A \cup D \subseteq \omega^{(o)}$. Let f be a one-one recursive function which enumerates $A \cup D$ and let $\{B_s : s < \omega\}$ be an effective enumeration of B. An effective enumeration $\{D_s : s < \omega\}$ of D is defined by $D_s = \{f(t) : t < s \text{ and } f(t) \in D\}$.

We construct r.e. sets A_0 and A_1 such that $\underset{\sim}{a}_0 = dg(A_0)$ and $\underset{\sim}{a}_1 = dg(A_1)$ have the desired properties.

To guarantee $\underset{\sim}{a} \leq \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{d} \leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$, we insure that $A \cup D = A_0^{(o)} \cup A_1^{(o)}$ and $D \subseteq A_j^{(o)}$ for $j \leq 1$. $A_0, A_1 \not\leq_T A \cup D$ ($\not\leq_T A$), and thus $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 \leq \underset{\sim}{a}$, is achieved by permitting.

To have $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$, we meet the requirements N_{2e+j} of the proof of theorem 5.2 (for all $e < \omega$, $j \leq 1$). The recursive length function l is defined as there. The restraint function r will be defined during the construction of A_0 and A_1.

As in the proof of theorem 2.2 fix a recursive function h s.t.

(5.48) $\forall e \, (W_e \cap \{x : D_x \subseteq \overline{D}\} \subseteq W_{h(e)})$

(5.49) $\forall e \, (W_e \cap \{x : D_x \subseteq \overline{D}\} \text{ finite} \Longrightarrow W_{h(e)} \text{ finite})$.

To guess the D-correctness of computations using the properties of h, a testset F will be needed whose r.e. index is known in advance (cf. the proof of theorem 2.2).

That $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist is guaranteed as in the proof of theorem 5.3 : We construct r.e. sets E_e and satisfy the requirements

$R_{\langle e,i\rangle}$ and the conditions (5.39) and (5.40) for all $e,i < \omega$. Also as in that proof is defined when $R_{\langle e,i\rangle}$ requires attention.

Let $\lambda x,y([x]_y)$ be a recursive function such that $W_{[x]_y} = \{z : z < \omega \text{ and } \langle y,z\rangle \in W_x\}$ for all $x,y < \omega$.

We can now describe the n-construction ($n < \omega$):

Stage s=0. For all $f < \omega$, $r(f,0) = R(f,0) = 0$.

Stage s+1. The stage consists of two steps. Numbers are put into A_0, A_1 and E_e only during the first step.

Step 1. We distinguish two cases.

Case 1 : $f(s) \in D$. Put $f(s)$ into A_0 and A_1. Set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } R(f,s) \leq f(s) \\ 0 & \text{otherwise.} \end{cases}$$

Case 2 : $f(s) \in A$. This case consists of two steps.

Substep 1. For every requirement $R_{\langle e,i\rangle}$ which requires attention at stage s+1 fix that number $\langle x,t,u,v\rangle$ through which $R_{\langle e,i\rangle}$ requires attention and for which v is minimal. Let w be the canonical index of the set $\{z : z < v \text{ and } z \notin D_s\}$. Put $\langle 2\langle e,i\rangle,w\rangle$ into F and simultaneously enumerate $W_{h([n]_{2\langle e,i\rangle})}$ and D until either w shows up in $W_{h([n]_{2\langle e,i\rangle})}$ or an element of D_w shows up in D. In the first case call $\langle x,t,u,v\rangle$ a correct attacker of $R_{\langle e,i\rangle}$ at stage s+1. (If none of both happens, step s+1 is never completed and the construction consists of only s+1 steps.)

Substep 2. Let y' be the least number y such that $f(s) < \max\{r(y,s),R(y,s)\}$ and let z' be the least index of a requirement which has a correct attacker at stage s+1.

If such a number y' (z') doesn't exist let $y' = \omega$ ($z' = \omega$).

Subcase 2.1 : $z' \leq y'$, $z' < \omega$. Let $z' = \langle e,i \rangle$ and let $\langle x,t,u,v \rangle$ be the correct attacker of $R_{\langle e,i \rangle}$. Put $\langle i,x \rangle$ into E_e, $\langle e+1,i,x,t,v \rangle$ into A_1, and $f(s)$ into A_0. Set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f < z' \\ v & \text{if } f = z' \\ 0 & \text{if } f > z'. \end{cases}$$

Subcase 2.2 : $y' < z'$. Let $y' = 2e+j$, $j \leq 1$. Put $f(s)$ into A_{1-j} and set

$$R(f,s+1) = \begin{cases} R(f,s) & \text{if } f \leq y' \\ 0 & \text{otherwise.} \end{cases}$$

Subcase 2.3 : $y' = z' = \omega$. Put $f(s)$ into A_0 and set $R(f,s+1) = R(f,s)$ for all $f < \omega$.

Step 2. For $f > s$ set $r(f,s+1) = 0$. For $f \leq s$, $f = 2e+j$, $j \leq 1$, and for all $y \leq l(f,s+1)$ let $u(y)$ be the canonical index of the set

$$\{ z : z < \max\{ u(e,A_{j,s+1},v,s+1) : v < y \} \text{ and } z \notin D_{s+1} \}.$$

Put $\langle 2f+1,u(y) \rangle$ into F and simultaneously enumerate $W_{h(\lceil n \rceil_{2f+1})}$ and D until either $u(y)$ shows up in $W_{h(\lceil n \rceil_{2f+1})}$ or an element of $D_{u(y)}$ shows up in D. (As in substep 1 none of both may happen and stage $s+1$ may never be completed).

If y_0 is the largest $y \leq l(f,s+1)$ s.t. $u(y)$ shows up in $W_{h(\lceil n \rceil_{2f+1})}$ then set $r(f,s+1) = \max\{ u(e,A_{j,s+1},v,s+1) : v \leq y_0 \}$. If no such y_0 exists, set $r(f,s+1) = 0$.

The n-construction is effective uniformly in n, $n < \omega$. Hence, by the recursion theorem, we can fix an n for the following such that

$F = W_n$ (for the set F enumerated in the n-construction). By (5.48) follows that every stage of the n-construction is completed. We show that the sets A_0 and A_1 constructed in the fixed n-construction have the desired properties.

Lemma 5.13. $\underset{\sim}{a} = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{d} \leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$.

Proof. By construction $A \cup D = A_0^{(o)} \cup A_1^{(o)}$ and $D \subseteq A_0^{(o)}, A_1^{(o)}$. Since $A \cap D = \emptyset$, this easily implies that $\underset{\sim}{a} \leq \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{d} \leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$. Since only numbers $\geqslant f(s)$ are put into A_0 and A_1 at stage s+1, $A_0, A_1 \leqq_T A \cup D$ ($\equiv_T A$), and thus $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 \leq \underset{\sim}{a}$.

Lemma 5.14. For all $f < \omega$:

(a) If $f = 2e+j$, $j \leq 1$, then the set

$$I_f = \{ s : \exists x \, (x \in A_{j,s+1} \setminus (A_{j,s} \cup D) \text{ and } x < r(f,s) \}$$

is finite.

(b) N_f is met.

(c) $\lim_s r(f,s) < \omega$ exists.

(d) R_f has a correct attacker at only finitely many stages.

(e) $\lim_s R(f,s) < \omega$ exists.

Proof by simultaneous induction on f. Parts (a),(b) and (c) are seen as the analogous claims in lemma 2.6. To see (d), fix $f = \langle e,i \rangle$ and by induction hypothesis choose a stage s_0 such that no requirement $R_{f'}$, where $f' < f$, has a correct attacker at a stage $\geqslant s_0$ and s.t. for all $s \geqslant s_0$

$$f(s) > \max \{ \max \{ r(f',s), R(f',s) \} : f' < f \}.$$

Then at any stage $s+1 \geqslant s_0$ at which R_f has a correct attacker, R_f receives attention. Hence, if we assume for a contradiction that this happens infinitely often, for every correct attacker $\langle x,t,u,v \rangle$ of R_f at stage $s+1 \geqslant s_0$, we have $D_s[v] \neq D[v]$. Otherwise, by choice of s_0, $R(f,s') > 0$ for all $s' > s$ and hence, by

(5.41), R_f never requires attention after stage s+1. It follows that the set $\{ y : \langle 2f,y\rangle \in F \} \cap \{ x : \mathbb{D}_x \subseteq \overline{D} \}$ is finite, while $W_{h([n]_{2f})}$ is infinite. Since $W_{[n]_{2f}} = \{ y : \langle 2f,y\rangle \in F \}$, this contradicts (5.49). +)

By lemma 5.14(b), $\underset{\sim}{b} \not\leq \underset{\sim}{a}_0, \underset{\sim}{a}_1$. It remains to show that $\underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist. For this purpose lemmata 5.11 and 5.12 are proved as in the proof of theorem 5.3.(In the proof of lemma 5.11 in the definition of s' replace the condition that $R_{\langle e,i\rangle}$ doesn't require attention after stage s' by the condition that $R_{\langle e,i\rangle}$ has no correct attacker after stage s', and derive as before a contradiction to this property of s'.)

Remark. As in the proof of theorem 2.2 one can see that the above constructed sets A_0 and A_1 are low. Hence in theorem 5.4 (corollary 5.4) we can assume that $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ ($\underset{\sim}{c}$ and $\underset{\sim}{d}$) are low.

+) (e) follows from (d) immediately.

§ 6 LATTICE EMBEDDINGS INTO THE R.E. DEGREES WHICH PRESERVE THE GREATEST ELEMENT.

The first embedding of a nonlinear lattice into $\underset{\sim}{R}$ is Lachlan's [4] and Yates'[33] minimal-pair construction, i.e. the embedding of the four-element Boolean algebra (diamond) into the r.e. degrees by a map which preserves the least element. Thomason [32] and Lachlan [5] extended the construction to show that every countable distributive lattice is embeddable preserving least element.

The case of the nondistributive lattices proved to be much more complicated. Lachlan [5] showed the embeddability of the two nondistributive five-element lattices. In [8] , however, Lachlan and Soare present a lattice which cannot be embedded into $\underset{\sim}{R}$. A characterization of the in $\underset{\sim}{R}$ embeddable lattices is not known.

While all known embeddings into $\underset{\sim}{R}$ automatically preserve the least element, for a longer time no (nontrivial) embeddings were known which preserve greatest element. Still in [28] it is conjectured that there is no embedding of the diamond into $\underset{\sim}{R}$ which preserves greatest element. Shoenfield and Soare [19] , and independently Lachlan [7] , finally refuted this conjecture and produced an embedding of the diamond into $\underset{\sim}{R}$ which preserved the greatest element.

We extend their result and show that every countable distributive lattice is embeddable into the r.e. degrees preserving greatest element (Corollary 6.1). These embeddings can be done simultaneously into the (effectively) noncappable degrees (Corollary 6.3).

In the case of the nondistributive lattices we show that the nonmodular five-element lattice and some lattices of similar struc-

ture can be embedded into $\mathcal{R}$ by a map which preserves the greatest element (Theorem 6.2 and Corollary 6.5).

Definition 6.1. Let $L = L(\cup, \cap, \leq)$ be a lattice.

(a) A map f from L into $\mathcal{R}$ is an embedding of L into the r.e. degrees if f is one-one and the following two conditions hold:

(6.1) $\forall a,b \in L$ ($f(a \cup b) = f(a) \cup f(b)$)

(6.2) $\forall a,b \in L$ ($f(a \cap b) = f(a) \cap f(b)$).

(b) An embedding $f : L \to \mathcal{R}$ preserves least (greatest) element if (6.3) ((6.4)) holds:

(6.3) If L has a least element (in respect to $\leq$), say 0, then $f(0) = \underset{\sim}{0}$.

(6.4) If L has a greatest element, say 1, then $f(1) = \underset{\sim}{0}'$.

In the following let $\mathcal{L} = \{ \alpha_i : i < \omega \} \subseteq \mathcal{P}(\omega)$ be a uniformly recursive sequence of recursive sets (i.e. the set $\{ \langle x,i \rangle : x \in \alpha_i \}$ is recursive) which forms an atomless Boolean algebra under $\cup$ (uninon), $\cap$ (intersection) and $^-$ (complement), contains ω and has $\emptyset$ as its only finite member. Further we may assume that $\forall i,j$ ($i \neq j \Rightarrow \alpha_i \neq \alpha_j$).

The following facts on $\mathcal{L}$ from lattice theory will be used:

(6.5) If $\alpha \in \mathcal{L}$, $\alpha \neq \emptyset, \omega$, then $\mathcal{L}[\geq \alpha] = \{ \beta \in \mathcal{L} : \beta \supseteq \alpha \}$ is isomorphic to $\mathcal{L}$.

(6.6) Every countable distributive lattice L is embeddable into $\mathcal{L}$ by a map which preserves greatest (least) element. If $|L| \geq 2$, greatest and least elements can be preserved.

Theorem 6.1. The Boolean algebra $\mathcal{L}$ is embeddable into the r.e. degrees by a map which preserves greatest element.

Corollary 6.1. Every countable distributive lattice is embeddable into $\underset{\sim}{R}$ by a map which preserves greatest element.

Proof. This follows from the theorem and (6.6).

Corollary 6.2. The Boolean algebra $\mathcal{L}$ is embeddable into the (effectively) noncappable degrees preserving greatest element.

Proof. By theorem 6.1 let f be an embedding of $\mathcal{L}$ into $\underset{\sim}{R}$ which preserves greatest element. Choose $\alpha \in \mathcal{L}$ s.t. $\alpha \neq \omega, \emptyset$. By (6.1), $f(\alpha) \cup f(\overline{\alpha}) = \underset{\sim}{0}'$, and thus, by (4.20), w.l.o.g. $f(\alpha) \in$ $\in$ ENC. Hence, by proposition 2.2, the restriction of f to $\mathcal{L}[\geq \alpha]$ is an embedding of $\mathcal{L}[\geq \alpha]$ into ENC which preserves greatest element. By (6.5), this implies corollary 6.2.

Corollary 6.3. Every countable distributive lattice is embeddable into ENC (NC) by a map which preserves greatest element.

Proof. This follows from corollary 6.2 and (6.6).

Corollary 6.4. A countable distributive lattice L is embeddable into ENC_0 by a map which preserves greatest and least elements iff $|L| \geq 2$ and, if a least element of L, say 0, exists, then

$$\forall a,b \in L \; (\; a \cap b = 0 \implies a = 0 \text{ or } b = 0 \;).$$

Proof. This is true by corollary 6.3 and proposition 2.2(c).

Proof of theorem 6.1. The proof builds on Thomason [32] and Shoenfield and Soare's embedding of the diamond by a map which preserves greatest element (cf. [19], unpublished).

We will construct uniformly r.e. sets A_i, $i<\omega$, which are pairwise disjoint, and an r.e. set B. We define $A_\alpha = \bigcup_{i\in\alpha} A_i$ for $\alpha\in\mathcal{L}$, and we want to show that the map f from $\mathcal{L}$ into $\underset{\sim}{R}$ defined by $f(\alpha) = dg(A_\alpha \oplus B)$, for $\alpha\in\mathcal{L}$, is an embedding which preserves greatest element.

By making the sets A_i pairwise disjoint, we immediately have (for $\alpha,\beta\in\mathcal{L}$):

(6.7) $dg(A_{\alpha\cup\beta}\oplus B) = dg(A_\alpha\oplus B)\cup dg(A_\beta\oplus B)$,

(6.8) $\alpha\subseteq\beta \implies dg(A_\alpha\oplus B)\leq dg(A_\beta\oplus B)$, and

(6.9) $dg(A_{\alpha\cap\beta}\oplus B)\leq dg(A_\alpha\oplus B), dg(A_\beta\oplus B)$.

Hence, to make f an embedding, it suffices to meet for all $e<\omega$ and $\alpha,\beta\in\mathcal{L}$ the requirements

R_e : $A_{(e)_0} \neq \{(e)_1\}^B$

$N_{\langle e,\alpha,\beta\rangle}$: $\{e\}^{A_\alpha\oplus B} = \{e\}^{A_\beta\oplus B}$ total $\implies$

$\implies$ $\{e\}^{A_\alpha\oplus B}$ recursive in $A_{\alpha\cap\beta}\oplus B$.

(In the index $\langle e,\alpha,\beta\rangle$ we identify α with i if $\alpha=\alpha_i$ etc. ; i.e. $\langle e,\alpha,\beta\rangle$ is a coded triple of numbers.)

The negative requirements $N_{\langle e,\alpha,\beta\rangle}$ imply, for any function h,

$$h\leq_T A_\alpha\oplus B, A_\beta\oplus B \implies h\leq_T A_{\alpha\cap\beta}\oplus B$$

(cf. the proof of theorem 1.1), and thus, by (6.9)

(6.10) $dg(A_{\alpha\cap\beta}\oplus B) = dg(A_\alpha\oplus B)\cap dg(A_\beta\oplus B)$.

The requirements R_e guarantee

(6.11) $\forall i\ (A_i\not\leq_T B)$.

A consequence of (6.10) and (6.11) is

(6.12) $dg(A_\alpha\oplus B)\leq dg(A_\beta\oplus B) \implies \alpha\subseteq\beta$.

(Proof: Assume for a contradiction that $dg(A_\alpha \oplus B) \leq dg(A_\beta \oplus B)$ but $\alpha \not\leq \beta$, say $i \in \alpha \setminus \beta$. Then $dg(A_i \oplus B) \leq dg(A_\alpha \oplus B) \leq$ $\leq dg(A_\beta \oplus B)$ (by assumption) and $dg(A_i \oplus B) \leq dg(A_{\overline{\beta}} \oplus B)$ (since $i \in \overline{\beta}$). Then, by (6.10), $dg(A_i \oplus B) \leq dg(A_{\beta \cap \overline{\beta}} \oplus B) = dg(B)$, which contradicts (6.11).)

By (6.12) the function f is one-one, and thus, by (6.7) and (6.10), it is an embedding of $\mathfrak{L}$. It remains to guarantee that f preserves the greatest element, i.e. that

$$f(\omega) = dg(A_\omega \oplus B) = \underset{\sim}{0}'.$$

This is done by coding markers. Fix a complete r.e. set K and a recursive function g which enumerates K without repetitions. Then coding markers Γ_e, $e < \omega$, are used such that

(6.13) $\forall s \; (\; \gamma(g(s),s) \neq \gamma(g(s),s+1) \;)$,

(6.14) $\forall x \; (\; \lim_s \gamma(x,s) < \omega \;$ exists $)$, and

(6.15) $\forall x \; (\; \gamma(x,s+1) \neq \gamma(x,s) \implies$

$$\exists y \leq \gamma(x,s) \; (\; y \in A_{\omega,s+1} \setminus A_{\omega,s} \text{ or } y \in B_{s+1} \setminus B_s \;)),$$

where $\gamma(e,s)$ denotes the position of the marker Γ_e at the end of stage s (of the subsequent construction).

Since the function $\lambda x,s(\gamma(x,s))$ will be recursive (by the effectiveness of the construction), (6.13) through (6.15) will imply that $K \leq_T A_\omega \oplus B$, and thus that $f(\omega) = \underset{\sim}{0}'$.

To the requirements $N_{\langle e,\alpha,\beta\rangle}$ a recursive length function l is defined by

$$l(\langle e,\alpha,\beta\rangle,s) = \max\{ x : \forall y < x \; (\; \{e\}^{A_\alpha,s} \oplus B_s \; (y) = \{e\}^{A_\beta,s} \oplus B_s \; (y) \downarrow \;) \; .$$

Based on this definition a recursive restraint function r and e-maximal stages are defined as in the proof of theorem 1.1.

The requirement R_e is attacked at (any) stage s+1 by x =

$= \gamma(\langle e, r(e,s)\rangle, s)$, i.e. we try to guarantee that $A_{(e)_0}(x) \neq \{(e)_1\}^B(x)$. Since $\limsup_s r(e,s)$ can be infinite, a single requirement may be attacked infinitely often (by infinitely many attackers). We will be successful with the attacker $x = \lim_s \gamma(\langle e, \liminf_s r(e,s)\rangle, s)$.

The number x requires attention at stage s+1, if there exists an $e < \omega$ such that $x = \langle e, r(e,s)\rangle$, $\gamma(x,s) \notin A_{(e)_0,s}$, $\{(e)_1\}^B_s(\gamma(x,s)) = 0$, and $x < g(s)$.

We can now describe the construction:

Stage s=0. For all $x < \omega$, $\gamma(x,0) = \langle 0, x\rangle$.

Stage s+1. We distinguish the following two cases:

Case 1 : No number requires attention at stage s+1. Then put $\gamma(g(s),s)$ into B and set

$$\gamma(y,s+1) = \begin{cases} \gamma(y,s) & \text{if } y < g(s) \\ \langle s+1, y\rangle & \text{otherwise.} \end{cases}$$

Case 2 : Otherwise. Let x be the least number which requires attention at stage s+1. Fix $e < \omega$ s.t. $x = \langle e, r(e,s)\rangle$. Put $\gamma(x,s)$ into $A_{(e)_0}$ and define

$$\gamma(y,s+1) = \begin{cases} \gamma(y,s) & \text{if } y \leq x \\ \langle s+1, y\rangle & \text{otherwise.} \end{cases}$$

We say that x receives attention at stage s+1.

The construction is effective and thus the sets A_i, $i < \omega$, and B are uniformly r.e. and the function $\lambda x,y(\gamma(x,y))$ is recursive. To show that the sets have the wanted properties, we prove the following lemmata.

Lemma 6.1. For all $x,y,s,t < \omega$:

(a) $\exists s' \leq s\ (\gamma(x,s) = \langle s', x\rangle)$

(b) $x \neq y \implies \gamma(x,s) \neq \gamma(y,t)$

(c) $x \leq \gamma(x,s) \leq \gamma(x,s+1)$

(d) $x \leq y \iff \gamma(x,s) \leq \gamma(y,s)$

(e) $\lim_s \gamma(x,s) < \omega$ exists.

Proof. The proof of (a) - (d) is trivial . (e) is proved by induction on x. For the inductive step fix x. By construction, for all $s < \omega$:

$$\gamma(x,s+1) \neq \gamma(x,s) \implies \text{(i) } \exists y < x \ (\ \gamma(y,s) \in A_{\omega,s+1} \setminus A_{\omega,s} \ ) \text{ or}$$
$$\text{(ii) } g(s) \leq x.$$

By induction hypothesis, (i) holds for at most finitely many s, and, since the function g is one-one, (ii) holds for only finitely many s. This implies (e).

Set $\gamma(x) = \lim_s \gamma(x,s)$.

An immediate consequence of lemma 6.1(b) and the choice of the attackers on the requirements R_e is that the sets A_i, $i < \omega$, are pairwise disjoint.

Lemma 6.2. $K \leq_T A_\omega \oplus B$.

Proof. It suffices to show (6.13)-(6.15). (6.13) and (6.15) follow from the construction immediately. (6.14) is true by lemma 6.1(e).

Lemma 6.3. For all $e < \omega$, $\liminf_s r(e,s) < \omega$.

Proof. This is proved as lemma 1.1.

Set $r(e) = \liminf_s r(e,s)$ and $S_e = \{ s : r(e,s) = r(e) \}$.

Lemma 6.4. Every number requires attention at most finitely often.

Proof by induction. For the inductive step fix x and, by induction hypothesis, choose a stage s_0 after which no number less than x requires attention. By lemma 6.1(e), choose $s_1 \geq s_0$ s.t.

$\forall s \geq s_1 \; (\; \gamma(x) = \gamma(x,s) \;).$

If x requires attention at a stage $s_2 \geq s_1$, then x receives attention at this stage and $\gamma(x,s_2) = \gamma(x)$ is enumerated in $A_{((x)_0)_0}$. Hence, by choice of s_1, x doesn't require attention after stage s_2.

Lemma 6.5. For all $e < \omega$, R_e is met.

Proof. Fix $e < \omega$, $e = \langle e_0, e_1 \rangle$. We show

$$\gamma(\langle e, r(e) \rangle) \in A_{e_0} \iff \{e_1\}^B(\gamma(\langle e, r(e) \rangle)) = 0.$$

"$\Longrightarrow$". Assume that $\gamma(\langle e, r(e) \rangle) \in A_{e_0}$ and let $x = \langle e, r(e) \rangle$. Then there is a stage s such that x receives attention at stage s+1 and $\gamma(x) = \gamma(x,s)$. I.e. $r(e,s) = r(e)$,

$0 = \{e_1\}_s^{B_s}(\gamma(x)) = \{e_1\}_{s+1}^{B_{s+1}}(\gamma(x))$ (note that no number enters B at stage s+1), $\gamma(x) \in A_{e_0,s+1}$, and

$\forall y > x \; (\; \gamma(y,s+1) = \langle s+1, y \rangle > s+1 \;).$

Hence the computation $\{e_1\}_{s+1}^{B_{s+1}}(\gamma(x))$ can be destroyed only if

(6.16) $\exists t > s \exists y \leq x \; (\; \gamma(y,t) \in B_{t+1} \setminus B_t \;).$

If (6.16) holds, fix $t > s$ and $y \leq x$ s.t. $\gamma(y,t) \in B_{t+1} \setminus B_t$. By construction, $y = g(t)$ and $\forall z \geq y \; (\; \gamma(z,t+1) = \langle t+1, z \rangle \;)$. This implies, by lemma 6.1(c), that $\gamma(x) \geq \langle t+1, x \rangle$. By lemma 6.1 (a) this contradicts the assumption that $\gamma(x) = \gamma(x,s)$ $(\leq \langle s, x \rangle)$. Hence (6.16) fails, and therefore the computation $\{e_1\}_{s+1}^{B_{s+1}}(\gamma(x))$ is B-correct, i.e. $\{e_1\}^B(\gamma(x)) = 0$.

"$\Longleftarrow$". Assume for a contradiction that $\{e_1\}^B(\gamma(\langle e, r(e) \rangle)) = 0$ and $\gamma(\langle e, r(e) \rangle) \notin A_{e_0}$. Then there are infinitely many stages s s.t. $\gamma(\langle e, r(e,s) \rangle, s) = \gamma(\langle e, r(e) \rangle)$ and $\langle e, r(e) \rangle$ requires attention at stage s+1. By lemma 6.4, $\langle e, r(e) \rangle$ eventually receives attention at such a stage s+1. Hence $\gamma(\langle e, r(e) \rangle)$ is enumerated into A_{e_0} contrary to our assumption.

Lemma 6.6. For all $e, f, s < \omega$:

(a) $e \leq f \Rightarrow r(e,s) \leq r(f,s)$ and $r(e) \leq r(f)$.

(b) $e < f \Rightarrow \exists s_0 \forall s \geq s_0 \; (s \in S_f \Rightarrow s \in S_e)$.

Proof. Part (a) follows from the definition of r immediately. To see (b), let s_0 be the least stage s such that

(i) $s > r(f)$,

(ii) $s \in S_e$, and

(iii) $\forall t \geq s \; (r(e,t) \geq r(e))$.

We claim that s_0 is a witness for (b): Choose any $s \geq s_0$ s.t. $s \notin S_e$. By (iii), $r(e,s) > r(e) = r(e,s_0)$. Hence, by definition of r, $r(e+1,s) \geq s_0$. With (a) follows that $r(f,s) \geq s_0$, and thus $r(f,s) > r(f)$ by (i); i.e. $s \notin S_f$.

By convention, let

$\forall s \; (r(-1,s) = 0)$ and $S_{-1} = \omega$.

For the following fix, for every $e < \omega$, a stage s_e such that

(6.17) $s_e \in S_{e-1}$,

(6.18) $\forall s \geq s_e \forall f \leq e-1 \; (s \in S_{e-1} \Rightarrow s \in S_f)$,

(6.19) $\forall s \geq s_e \forall f \leq e \; (r(f,s) \geq r(f))$, and

(6.20) $A_{\omega,s_e}[\gamma(\langle e,r(e)\rangle)+1] = A_\omega[\gamma(\langle e,r(e)\rangle)+1]$.

Such a stage exists by lemmata 6.3 and 6.6.

Lemma 6.7. Let s and t be stages s.t. $s_e \leq s < t$, s is e-maximal and in S_{e-1}, and t is not e-maximal or no element of S_{e-1}. Then at stage t+1 no number $x < s$ is enumerated into A_ω (for all e,s, $t < \omega$).

Proof. Let $x \in A_{\omega,t+1} \setminus A_{\omega,t}$. Then $x = \gamma(\langle f,r(f,t)\rangle,t)$ for some $f < \omega$.

Case 1 : $f \geq e$. By choice of s and t and by (6.19), $r(e,t) \geq s$, and thus (by lemma 6.6(a)) $r(f,t) \geq s$. It follows that $x \geq s$.

Case 2 : $f < e$. Since, for all $f \leq e$ and $t < \omega$, $\gamma(\langle f,r(f)\rangle,t) \leq \gamma(\langle e,r(e)\rangle)$ (by lemma 6.1), it follows from (6.18) - (6.20) that $r(f,t) > r(f) = r(f,s)$.

Subcase 2.1 : $r(f-1,s) \neq r(f-1,t)$. By (6.18) and (6.19), $r(f-1) = r(f-1,s) < r(f-1,t)$. By definition of r, $r(f,t) \geq s$ and thus $x \geq s$.

Subcase 2.2 : $r(f-1,s) = r(f-1,t)$. By (6.18), $s,t \in S_{f-1}$. Hence, since $r(f,t) > r(f,s)$, t is not f-maximal and there is an f-maximal stage $s' \in S_{f-1}$ s.t. $s \leq s' < t$. Hence $r(f,t) \geq s' \geq s$ and therefore $x \geq s$.

Lemma 6.8. For all $f < \omega$, $\alpha,\beta \in \mathcal{G}$, $N_{\langle f,\alpha,\beta\rangle}$ is met.

Proof. Fix $e = \langle f,\alpha,\beta\rangle$, and assume that

$$h = \{f\}^{A_\alpha} \oplus B = \{f\}^{A_\beta} \oplus B$$

is a total function. Then $\lim_s l(e,s) = \omega$.

We have to show that h is recursive in $A_{\alpha\cap\beta} \oplus B$.

For all x and s s.t. $x < l(e,s)$ let

$h_s(x) = \{f\}_s^{A_\alpha,s} \oplus B_s(x) = \{f\}_s^{A_\beta,s} \oplus B_s(x)$,

$u_s^\alpha(x) = u(f,A_{\alpha,s} \oplus B_s,x,s)$, and

$u_s^\beta(x) = u(f,A_{\beta,s} \oplus B_s,x,s)$.

To compute h(p) recursively in $A_{\alpha\cap\beta} \oplus B$, for given $p < \omega$, let s_0 be the least stage s such that $s \geq s_e$,

(6.21) $s \in S_{e-1}$ and s is e-maximal,

(6.22) $l(e,s) > p$, and

(6.23) $A_{\alpha\cap\beta,s} \oplus B_s[\min\{u_s^\alpha(p),u_s^\beta(p)\}] = A_{\alpha\cap\beta} \oplus B[\min\{u_s^\alpha(p),u_s^\beta(p)\}]$

Such a stage s_0 exists since, by $\lim_s l(e,s) = \omega$, (6.21) holds for infinitely many and (6.22) for almost all stages s, and (6.23) holds for almost all s by the assumption that h is total.

We claim: (+) $h(p) = h_{s_0}(p)$.

Since s_0 can be computed recursively in $A_{\alpha\cap\beta} \oplus B$ (uniformly in p), it follows that h is recursive in $A_{\alpha\cap\beta} \oplus B$.

To see (+), it suffices to show for all $s \geqslant s_0$:

(6.24) If $s \in S_{e-1}$ and s is e-maximal, then (6.22) and (6.23) hold and $h_s(p) = h_{s_0}(p)$.

We prove (6.24) by induction on $s \geqslant s_0$: For $s = s_0$, (6.24) holds by choice of s_0. For the inductive step, fix $t > s_0$ s.t. $t \in S_{e-1}$ and t is e-maximal. Then $l(e,t) > p$ is immediate. Let s be the largest stage less than t s.t. $s \in S_{e-1}$ and s is e-maximal. By choice of s_0, $s \geqslant s_0$. Hence, by induction hypothesis, (6.24) holds for s. By symmetry, w.l.o.g. $u_s^\alpha(p) \leq u_s^\beta(p)$, and thus

(6.25) $A_{\alpha\cap\beta,s} \oplus B_s [u_s^\alpha(p)] = A_{\alpha\cap\beta} \oplus B [u_s^\alpha(p)]$.

Case 1 : $\{f\}_s^{A_\alpha,s} \oplus B_s (p)$ is destroyed at stage s+1. Then, by (6.25) a number $z < \frac{1}{2}\cdot u_s^\alpha(p)$ is enumerated into $A_{\alpha\cap\beta}$ at stage s+1 and $z = \gamma(v,s)$ for some $v < \omega$. Hence, for all $s' \geqslant s+1$ and $w < \omega$,

(6.26) $\gamma(w,s') \leq z$ or $\gamma(w,s') \geqslant s+1$.

By (6.25), no number $\leq z$ ($< \frac{1}{2}\cdot u_s^\alpha(p)$) can be enumerated in $A_{\alpha\cap\beta}$ or B after stage s, and therefore, by (6.26),

(6.27) $A_{\alpha\cap\beta,s} \oplus B_s [s+1] = A_{\alpha\cap\beta} \oplus B [s+1]$.

By lemma 6.7, at no stage t' s.t. $s+1 < t' \leq t$ a number less than s is enumerated in A_ω. Since at stage s+1 no number enters $A_{\beta-\alpha}$, it follows with (6.27) that

$\{f\}_t^{A_\beta,t} \oplus B_t (p) = \{f\}_s^{A_\beta,s} \oplus B_s (p)$ and $u_t^\beta(p) = u_s^\beta(p) \leq s$.

Hence, by (6.27), (6.23) holds for stage t = s, and, by induction hypothesis, $h_t(p) = h_s(p) = h_{s_0}(p)$. It follows that (6.24) holds for t (=s).

Case 2 : $\{f\}_s^{A_\alpha,s} \oplus B_s (p)$ is not destroyed at stage s+1. Then, by lemma 6.7 (since $u_s^\alpha(p) \leq s$), the computation cannot be destroyed before stage t+1. I.e.

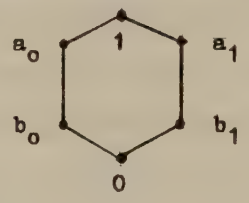
$\{f\}^{A_t^\alpha,t} \oplus B_t\,(p) = \{f\}^{A_s^\alpha,s} \oplus B_s\,(p)$ and $u_t^\alpha(p) = u_s^\alpha(p)$.

Hence, by (6.25) and induction hypothesis, (6.24) holds for t.

This completes the proof of lemma 6.8.

As we have seen already, the theorem follows from lemmata 6.2, 6.5 and 6.8.

Theorem 6.2. The following lattice can be embedded into the r.e. degrees preserving greatest element :



Corollary 6.5. Let $L = L(\cup, \cap, \leq)$ be a countable lattice s.t. L contains two sublattices L_0 and L_1 with the following properties:

(i) $L = L_0 \cup L_1 \cup \{0,1\}$, where 0 and 1 are the least and greatest elements of L, respectively.

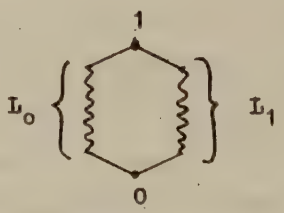
(ii) $\forall a \in L_0 \cup L_1\ (0 < a < 1)$.

(iii) $\forall a \in L_0 \forall b \in L_1\ (a \not\leq b$ and $b \not\leq a)$.

(iv) L_0 and L_1 are linear orderings (in respect to $\leq$).

Then L can be embedded into the r.e. degrees preserving greatest element.

(Diagram of L :



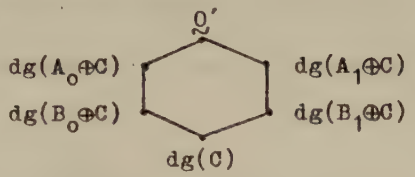
)

Proof. This is an immediate consequence of theorem 6.2 and the

density theorem.

Proof of theorem 6.2. We will construct r.e. sets A_0, A_1, B_0, B_1 and C whose degrees satisfy

(6.28)



Let K be a complete set and g be a one-one recursive function which enumerates K. Then, to show (6.28), it suffices to satisfy the following conditions:

(6.29) $K \leq_T A_0 \oplus A_1 \oplus C$,

(6.30) $A_0, A_1 \leq_T B_0 \oplus B_1$,

(6.31) $\forall i \leq 1 \; (B_i \leq_T A_i)$,

(6.32) $\forall i \leq 1 \; (A_i \not\leq_T B_i \oplus C)$,

(6.33) $\forall i \leq 1 \; (B_i \not\leq_T C)$, and

(6.34) $dg(A_0 \oplus C) \cap dg(A_1 \oplus C) = dg(C)$.

Condition (6.29) is guaranteed by markers. I.e. a recursive function $\lambda x,y(\gamma(x,y))$ will be defined during the construction which satisfies conditions (6.13),(6.14) (in the proof of theorem 6.1) and

(6.35) $\forall x,s \; (\; \gamma(x,s) \neq \gamma(x,s+1) \implies$

$\implies \exists y \leq \gamma(x,s) \; (\; y \in (A_{0,s+1} \setminus A_{0,s}) \cup (A_{1,s+1} \setminus A_{1,s}) \cup$

$\cup (C_{s+1} \setminus C_s) \;) \;)$.

To get (6.30) we use markers too. A recursive function $\lambda x,y(\delta(x,y))$ will be defined such that

(6.36) $\forall x,s \; (\; (\delta(x,s+1) \neq \delta(x,s)$ or $\exists i \leq 1 \; (\; x \in A_{i,s+1} \setminus A_{i,s}))$

$\implies \exists y \leq \delta(x,s) \exists i \leq 1 \; (\; y \in B_{i,s+1} \setminus B_{i,s} \;) \;)$, and

(6.37) $\lim_s \delta(x,s) < \omega$ exists.

To see that (6.36) and (6.37) imply (6.30), fix $p < \omega$. By (6.36) and (6.37), recursively in $B_0 \oplus B_1$ find a stage s_0 such that $\forall s \geq s_0 (\delta(p,s) = \delta(p,s_0))$ and $B_{0,s_0} \oplus B_{1,s_0}[\delta(p,s_0)] = B_0 \oplus B_1[\delta(p,s_0)]$. Then, for $i \leq 1$, $p \in A_i$ iff $p \in A_{i,s_0}$.
The condition (6.31) is achieved by the permitting method.

To insure (6.32) and (6.33) we will meet the requirements

R_{4e+i} : $A_i \neq \{e\}^{B_i} \oplus C$ and

R_{4e+2+i} : $B_i \neq \{e\}^C$

for all $e < \omega$ and $i < 2$.

Condition (6.34) is guaranteed by the requirements

N_e : $\{e\}^{A_0} \oplus C = \{e\}^{A_1} \oplus C$ total $\Rightarrow \{e\}^{A_0} \oplus C$ recursive in C

for all $e < \omega$.

A length function l is defined by

$$l(e,s) = \max \{ x : \forall y < x \, (\{e\}^{A_0,s}_s \oplus C_s(y) = \{e\}^{A_1,s}_s \oplus C_s(y) \downarrow) \}.$$

Based on this definition a restraint function r and e-maximal stages are defined as in the proof of theorem 1.1.

A number x requires attention at stage s+1 if there exist $e < \omega$ and $i < 4$ such that $x = \langle 4e+i, r(e,s) \rangle$, $x < g(s)$, and one of the following conditions holds:

(6.38)
$$\begin{cases} i \leq 1, \\ \{e\}^{B_{i,s}}_s \oplus C_s (\gamma(x,s)) = 0, \\ \gamma(x,s) \notin A_{i,s}, \text{ and} \\ (\delta(\gamma(x,s),s) \geq u(e, B_{i,s} \oplus C_s, \gamma(x,s), s) \text{ or} \\ \qquad \gamma(x,s) = \delta(\gamma(x,s),s) \notin B_{1-i,s}) \end{cases}$$

(6.39)
$$\begin{cases} i = 2+j \text{ for some } j \leq 1, \\ \{e\}^{C_s}_s (\gamma(x,s)) = 0, \text{ and} \\ \gamma(x,s) \notin A_{j,s}. \end{cases}$$

We can now describe the construction:

Stage s=0. For all $x < \omega$, $\chi(x,0) = \langle 0,x \rangle$ and $\delta(x,0) = x$.

Stage s+1. Distinguish the following two cases.

Case 1 : No number requires attention at stage s+1. Then put $\chi(g(s),s)$ into C and define (for all $y < \omega$):

$$\chi(y,s+1) = \begin{cases} \chi(y,s) & \text{if } y < g(s) \\ \langle s+1,y \rangle & \text{otherwise} \end{cases}$$

and $\delta(y,s+1) = \delta(y,s)$.

Case 2 : Otherwise. Then let x be the least number which requires attention at stage s+1.(We say that x receives attention at stage s+1). Fix $e < \omega$ and $i < 4$ such that $x = \langle 4e+i, r(e,s) \rangle$. Set

$$\chi(y,s+1) = \begin{cases} \chi(y,s) & \text{if } y \leq x \\ \langle s+1,y \rangle & \text{otherwise,} \end{cases}$$

and distinguish the following subcases.

Case 2.1.: $i \leq 1$ and $\delta(\chi(x,s),s) \geqslant u(e, B_{i,s} \oplus C_s, \chi(x,s),s)$. Then put $\chi(x,s)$ into A_i and $\delta(\chi(x,s),s)$ in both A_i and B_i. Set $\delta(y,s+1) = \delta(y,s)$ for all $y < \omega$.

Case 2.2 : $i \leq 1$ and case 2.1 doesn't hold. Then put $\delta(\chi(x,s),s) = \chi(x,s)$ into A_{i-1} and B_{i-1}. Define

$$\delta(y,s+1) = \begin{cases} \langle s+1,x \rangle & \text{if } y = \chi(x,s) \\ \delta(y,s) & \text{otherwise.} \end{cases}$$

Case 2.3 : $i = 2+j$ for some $j \leq 1$. Then put $\chi(x,s)$ into A_j and B_j and set, for all $y < \omega$, $\delta(y,s+1) = \delta(y,s)$.

The construction is effective, i.e. the constructed sets are r.e. and the functions $\lambda x,s(\chi(x,s))$ and $\lambda x,s(\delta(x,s))$ are recursive. The subsequent lemmata will show that the sets A_i, B_i and C ($i \leq 1$) have the desired properties.

As in the proof of theorem 6.1 it is seen that the function γ has the properties listed in lemma 6.1. In particular (6.13) holds. That (6.14) and (6.35) hold too follows from the construction easily. Hence $K \leq_T A_0 \oplus A_1 \oplus C$, i.e. condition (6.29) is satisfied.

Let $\gamma(x) = \lim_s \gamma(x,s)$ ($< \omega$) for all $x < \omega$.

Lemma 6.9. For all $x,y,s,t < \omega$:

(a) $x \leq \delta(x,s) \leq \delta(x,s+1)$

(b) ($x \neq y$, $x \neq \delta(x,s)$ and $y \neq \delta(y,t)$) $\Rightarrow \delta(x,s) \neq \delta(y,t)$

(c) $x \neq \delta(x,s) \Rightarrow \delta(x,s) \neq \gamma(y,t)$

(d) $x \neq \delta(x,s) \Rightarrow \delta(x,s) = \delta(x,s+1)$

(e) $\lim_s \delta(x,s) < \omega$ exists.

Proof. This follows easily from the construction and the conditions (6.38) and (6.39) in the definition of "require attention". Note that (e) follows from (d).

Let $\delta(x) = \lim_s \delta(x,s)$ for all $x < \omega$.

Lemma 6.10. For $i \leq 1$:

(a) $B_i \leq_T A_i$

(b) $A_i \leq_T B_0 \oplus B_1$.

Proof. To see (a) it suffices to show

(6.40) $\forall x,s$ ($x \in B_{i,s+1} \setminus B_{i,s} \Rightarrow x \in A_{i,s+1} \setminus A_{i,s}$).

If $x \in B_{i,s+1}$ then $x \in A_{i,s+1}$ since a number x is put into B_i only if at the same stage x is put into A_i. Hence it suffices to show for all $x,s < \omega$ that

$$x \in B_{i,s+1} \setminus B_{i,s} \Rightarrow x \notin A_{i,s}.$$

Assume that $x \in B_{i,s+1} \setminus B_{i,s}$. Then a number x' receives attention at stage s+1. In case 2.1 of the construction, $x = \delta(\gamma(x',s),s)$ and $\gamma(x',s) \notin A_{i,s}$. Hence w.l.o.g. $x \neq \gamma(x',s)$. Then, by lemma

6.9(c), x can be enumerated in A_i only via case 2.1, and thus only at a stage t where x is enumerated in B_i. It follows that $x \notin A_{i,s}$. In case 2.2, $x = \delta(\gamma(x',s),s) = \gamma(x',s) \notin A_{i,s}$ by (6.38). In case 2.3, $x = \gamma(x',s) \notin A_{i,s}$ by (6.39). Hence in any case $x \notin A_{i,s}$, which completes the proof of (6.40) and part (a) of the lemma.

To see (b) we have to show that (6.36) and (6.37) are true. (6.37) holds by lemma 6.9(e). (6.36) follows from the following facts, which can easily be derived from the construction: For $i \leq 1$,

(6.41) $\forall x,s \; (\; x \notin A_{i,s} \implies \delta(x,s) \notin B_{i,s}$ and $x \notin B_{i,s} \;)$,

(6.42) $\forall x,s \; (\; x \in A_{i,s} \implies \delta(x,s) \in B_{i,s}$ or $x \in B_{i,s} \;)$, and

(6.43) $\forall x,s \; (\; \delta(x,s) \neq \delta(x,s+1)$.

$x = \delta(x,s)$, $x \notin B_{0,s} \cup B_{1,s}$ and $x \in B_{0,s+1} \cup B_{1,s+1} \;)$.

Lemma 6.11. For all $e < \omega$, $\lim\inf_s r(e,s) < \omega$.

Lemma 6.11 is proved as lemma 1.1.

We define, for all $e < \omega$, $r(e) = \lim\inf_s r(e,s)$ and $S_e = \{ s : r(e,s) = r(e) \}$.

Note that if, for all $s \geqslant s_0$, $\gamma(x,s) = \gamma(x)$ then x can receive attention at most twice after stage s_0. (If it receives attention via case 2.1 at a stage $s_1+1 > s_0$ then $x = \langle 4e+i, r(e,s_1) \rangle$ for some $e < \omega$ and $i \leq 1$, and $\gamma(x) \in A_{i,s_1+1}$. Hence, by (6.38), x never requires attention after stage s_1+1. It is seen similarly that if x receives attention via case 2.3 at a stage $> s_0$ then it never requires attention afterwards. Finally, if x receives attention via 2.2 at a stage $s_1+1 > s_0$ then $\delta(\gamma(x),s_1+1) \neq \gamma(x)$ and thus x can receive attention only via case 2.1 after stage s_1+1.). It follows, by induction, that every number requires attention at most finitely often.

Lemma 6.12. For all $f < \omega$, the requirement R_f is met.

Proof. Fix $f,e < \omega$ and $i < 4$ such that $f = 4e+i$. Let $x = \langle f, r(e)\rangle$.

Case 1 : $i \leq 1$. We show that

$$\chi(x) \in A_i \iff \{e\}^{B_i \oplus C}(\chi(x)) = 0.$$

"$\Rightarrow$". Assume that $\chi(x) \in A_i$. Fix s such that $\chi(x) \in A_{i,s+1} \setminus A_{i,s}$. Then x receives attention at stage s+1 (via case 2.1). I.e. $r(e,s) = r(e)$, $\chi(x,s) = \chi(x)$, $\{e\}_s^{B_i,s \oplus C_s}(\chi(x)) = 0$,

(6.44) $d(\chi(x),s) \geqslant u(e, B_{i,s} \oplus C_s, \chi(x), s)$, and

(6.45) $\forall y > x\ (\chi(y,s+1) = \langle s+1, y\rangle \geqslant s+1)$.

By (6.44),

(6.46) $\{e\}_{s+1}^{B_i,s+1 \oplus C_{s+1}}(\chi(x)) = \{e\}_s^{B_i,s \oplus C_s}(\chi(x)) = 0$.

We claim that no number less than s+1 is enumerated in $B_i \oplus C$ after stage s+1, and thus the computation (6.46) is $(B_i \oplus C)$-correct, i.e. $\{e\}^{B_i \oplus C}(\chi(x)) = 0$.

Assume for a contradiction that $y < s+1$ is put into $B_i \oplus C$ at stage $t+1 > s+1$. By (6.45) (and lemmata 6.1(c) and 6.9(a)), $y = \chi(z,t)$ or $y = d(\chi(z,t),t)$ for some $z < x$ and z receives attention at stage t+1. Hence by lemma 6.1(a) and (c),

$$\chi(x) \geqslant \chi(x,t+1) \geqslant \langle t+1, x\rangle > \langle s, x\rangle \geqslant \chi(x,s) = \chi(x).$$

Contradiction.

"$\Leftarrow$". Assume for a contradiction that $\{e\}^{B_i \oplus C}(\chi(x)) = 0$ and $\chi(x) \notin A_i$. Distinguish the following two cases.

Case 1.1 : $\chi(x) \notin B_{1-i}$. Then, by (6.38),

$$\forall s\ (d(\chi(x),s) = \chi(x)).$$

Hence there are infinitely many stages s such that

$$\chi(x) = \chi(x,s),\ r(e,s) = r(e),\ \{e\}_s^{B_i,s \oplus C_s}(\chi(x)) = 0,$$
$$\chi(x) \notin A_{i,s} \text{ and } \chi(x) = d(\chi(x),s) \notin B_{1-i,s}.$$

Then x requires attention at every such stage s+1 and thus eventually receives attention at such a stage s+1. Hence either $\gamma(x) \in A_{i,s+1}$ or $\gamma(x) \in B_{1-i,s+1}$. In either case this contradicts our assumptions.

Case 1.2 : $\gamma(x) \in B_{1-i}$. Fix s such that $\gamma(x) \in B_{1-i,s+1} \setminus B_{1-i,s}$. Then x receives attention via case 2.2 at stage s+1. Then

$$\partial(\gamma(x),s+1) \geq s+1 \geq u(e,B_{i,s} \oplus C_s, \gamma(x), s) \quad \text{and}$$
$$\{e\}^{B_{i,s} \oplus C_s}(\gamma(x)) = 0,$$

and as in the proof of the direction " $\Rightarrow$ " one can see that this computation is $(B_i \oplus C)$-correct. I.e.

$$\forall t \geq s \ (\{e\}^{B_{i,t} \oplus C_t}(\gamma(x)) = 0 \quad \text{and} \quad u(e,B_{i,t} \oplus C_t, \gamma(x), t) \leq s).$$

Hence there are infinitely many stages t such that

$$\gamma(x) = \gamma(x,t), \quad r(e) = r(e,t), \quad \{e\}^{B_{i,t} \oplus C_t}(\gamma(x)) = 0,$$
$$\gamma(x) \notin A_{i,t}, \text{ and } \partial(\gamma(x),t) \geq u(e,B_{i,t} \oplus C_t, \gamma(x), t).$$

Since x requires attention at every such stage t+1, it eventually receives attention via case 2.1 at such a stage. It follows that $\gamma(x) \in A_i$ contrary to our assumption.

Case 2 : $i > 1$. Fix $j \leq 1$ s.t. $i = 2+j$. Then

$$\gamma(x) \in B_j \iff \{e\}^C(\gamma(x)) = 0.$$

This is proved as lemma 6.5.

As in the proof of theorem 6.1, lemma 6.6 holds. Hence for every $e < \omega$ we can choose an s_e which satisfies the conditions (6.17) through (6.19) (in the proof of theorem 6.1) and

(6.47) For $i \leq 1$ and $x = \gamma(\langle 4e+3, r(e)\rangle)+1$,

$$A_{i,s_e}[x] = A_i[x], \quad B_{i,s_e}[x] = B_i[x], \text{ and } \quad C_{s_e}[x] = C[x].$$

Lemma 6.13. Let $s,t,e < \omega$ such that $s_e < s < t$, $s \in S_{e-1}$ and s is e-maximal, and $t \notin S_{e-1}$ or t not e-maximal. Then at stage t+1 no number less than s is enumerated in A_i or B_i ($i \leq 1$).

Proof. Let x be enumerated in A_i or B_i ($i \leq 1$) at stage t+1. Then, for some $y < \omega$, $x = \gamma(y,t)$ or $x = \delta(\gamma(y,t),t)$. In the second case $x \geq \gamma(y,t)$ and $\gamma(y,t)$ is enumerated in A_i or B_i at stage t+1 too. Hence w.l.o.g. we may assume that $x = \gamma(y,t)$. Then the lemma is proved as lemma 6.7.

Lemma 6.14. For all $e < \omega$, N_e is met.

Proof. The proof is very similar to that of lemma 6.8. Assume that $h = \{e\}^{A_0 \oplus C} = \{e\}^{A_1 \oplus C}$ is a total function. We have to show that h is recursive in C.

For $x < l(e,s)$ let $h_s(x) = \{e\}_s^{A_{0,s} \oplus C_s}(x)$ and $u(i,x,s) = u(e,A_{i,s} \oplus C_s, x, s)$ ($i \leq 1$).

To compute h(p) recursively in C, for given $p < \omega$, let s_0 be the least stage $s \geq s_e$ such that

(6.48) $s \in S_{e-1}$ and s is e-maximal,

(6.49) $l(e,s) > p$, and

(6.50) $C_s[\min\{u(i,p,s) : i \leq 1\}] = C[\min\{u(i,p,s) : i \leq 1\}]$.

Such a stage exists (since by assumption $\lim_s l(e,s) = \omega$) and can be found recursively in C (uniformly in p).

We claim that $h_{s_0}(p) = h(p)$ (and thus $h \leq_T C$).

To see this it suffices to show for all $s \geq s_0$

(6.51) If s is an e-maximal stage in S_{e-1} then (6.49) and (6.50) hold and $h_s(p) = h_{s_0}(p)$.

(6.51) is proved by induction on $s \geq s_0$. For $s = s_0$ (6.51) holds by definition of s_0. For the inductive step fix $s > s_0$ such that $s \in S_{e-1}$ and s is e-maximal. Then (6.49) follows immediately from the choice of s_0. To see (6.50) let t be the greatest stage less than s s.t. $t \in S_{e-1}$ and t is e-maximal. By choice of s_0, $s_0 \leq t$. Hence, by induction hypothesis,

(6.52) $C_t[\min\{u(i,p,t) : i \leq 1\}] = C[\min\{u(i,p,t) : i < 1\}]$,

and

(6.53) $h_t(p) = h_{s_o}(p)$.

W.l.o.g. $u(0,p,t) \leq u(1,p,t)$. Hence, by (6.52),

(6.54) $C_t[u(0,p,t)] = C[u(0,p,t)]$.

Distinguish the following two cases.

Case 1 : $\{e\}^{A_t o,t} \oplus C_t(p)$ is destroyed at stage t+1. Then, by (6.54), a number $x < \frac{1}{2}\cdot u(0,p,t)$ is enumerated in A_o at stage t+1. Hence, for some $y \leq x$, $x = \chi(y,t)$ or $x = d(\chi(y,t),t)$ and

$\forall z > y\ (\ \chi(z,t+1) = \langle t+1,z\rangle\ )$.

It follows

(6.55) $\forall z \forall s' > t\ (\ \chi(z,s') \leq x$ or $\chi(z,s') \geq t+1\ )$.

Since only numbers of the form $\chi(z,s')$, where $s' > t$, are put into C after stage t, (6.54),(6.55) and $x < u(0,p,t)$ imply

(6.56) $C_t[t+1] = C[t+1]$.

Since no number is enumerated in A_1 at stage t+1 by construction, and since no number less than t enters A_1 at a stage s' such that $t+1 < s' < s+1$ (by lemma 6.13), it follows that

(6.57) $\{e\}^{A_t 1,t} \oplus C_t(p) = \{e\}^{A_s 1,s} \oplus C_s(p)$ and

$u(1,t,p) = u(1,p,s)$ ($\leq t$).

(6.53),(6.56) and (6.57) imply that (6.51) holds for s.

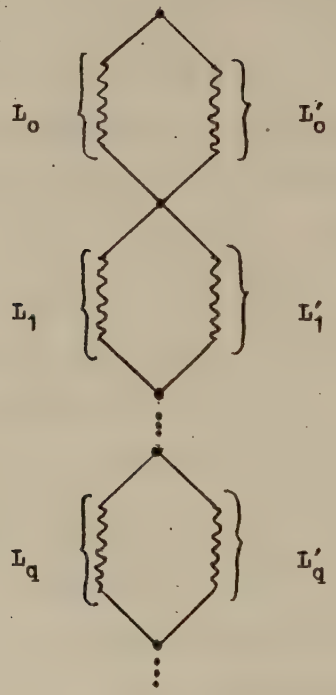
Case 2 : $\{e\}^{A_t o,t} \oplus C_t(p)$ is not destroyed at stage t+1. Then, by lemma 6.13,

$\{e\}^{A_t o,t} \oplus C_t(p) = \{e\}^{A_s o,s} \oplus C_s(p)$ and $u(0,p,t) = u(0,p,s)$.

This, together with (6.52) and (6.53), implies (6.51).

This completes the proof of lemma 6.14 and the proof of the theorem.

The proofs of the two theorems of this section can be combined to prove that some more nondistributive lattices are embeddable in $\underset{\sim}{R}$ preserving greatest element. E.g. this is the case for any countable "chain" of lattices as described in corollary 6.2 (cf the figure below).



§ 7 LATTICE EMBEDDINGS INTO THE R.E. DEGREES WHICH PRESERVE GREATEST AND LEAST ELEMENTS.

While every countable distributive lattice is embeddable into the r.e. degrees preserving greatest or least element (cf. Corollary 6.1 and Thomason [32]/ Lachlan [5], respectively), not every such lattice is embeddable by a map which preserves both greatest and least elements. Lachlan [4] showed that no embedding of the diamond (and hence of any Boolean algebra which has more than two members) preserves greatest and least elements. In this section we give a first positive result. We define a subclass of lattices (Definition 7.1) and show that every countable distributive member of it can be embedded into $\underset{\sim}{R}$ by a map which preserves both greatest and least elements (Corollary 7.1).

Definition 7.1. A lattice $L = L(\cup, \cap, \leq)$ is disconnected if there are $a, b \in L$ such that

(7.1) $a < b$,

(7.2) $\forall c \in L$ ($c \leq a$ or $a < c < b$ or $b \leq c$), and

(7.3) $\forall c, d \in L$ ($a < c, d < b \implies$ ($c \leq d$ or $d \leq c$)).

Otherwise L is connected.

For r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that $\underset{\sim}{a} < \underset{\sim}{b}$ let

$[\underset{\sim}{a}, \underset{\sim}{b}] = \{\underset{\sim}{c} : \underset{\sim}{a} \leq \underset{\sim}{c} \leq \underset{\sim}{b}\}$ and

$(\underset{\sim}{a}, \underset{\sim}{b}) = \{\underset{\sim}{c} : \underset{\sim}{a} < \underset{\sim}{c} < \underset{\sim}{b}\}$.

An embedding f of a lattice L into $\underset{\sim}{R}$ s.t. $f[L] \subseteq [\underset{\sim}{a}, \underset{\sim}{b}]$ ($f[L] \subseteq (\underset{\sim}{a}, \underset{\sim}{b})$) is an embedding of L into $[\underset{\sim}{a}, \underset{\sim}{b}]$ ($(\underset{\sim}{a}, \underset{\sim}{b})$). An embedding f of L into $[\underset{\sim}{a}, \underset{\sim}{b}]$ preserves greatest (least) element if the

following condition (7.4) ((7.5)) holds:

(7.4) If L has a greatest element, say 1, then $f(1) = \underset{\sim}{b}$.

(7.5) If L has a least element, say 0, then $f(0) = \underset{\sim}{a}$.

Let $\mathcal{L}$ be the Boolean algebra defined in § 6.

Theorem 7.1. There exist r.e. degrees $\underset{\sim}{b}$ and $\underset{\sim}{c}$ such that $\underset{\sim}{c} < \underset{\sim}{b}$ and there are embeddings of $\mathcal{L}$ into the intervals $[\underset{\sim}{0},\underset{\sim}{c}]$ and $[\underset{\sim}{b},\underset{\sim}{0}']$ which preserve greatest and least elements.

Corollary 7.1. Every countable disconnected distributive lattice is embeddable into the r.e. degrees preserving greatest and least elements.

Proof. By the theorem, fix $\underset{\sim}{b}$, $\underset{\sim}{c}$, f_0 and f_1 such that $\underset{\sim}{c} < \underset{\sim}{b}$, and f_0 and f_1 are embeddings of $\mathcal{L}$ into $[\underset{\sim}{0},\underset{\sim}{c}]$ and $[\underset{\sim}{b},\underset{\sim}{0}']$, respectively, which preserve greatest and least elements.
Let L be a countable disconnected distributive lattice. Fix $a,b \in L$ such that (7.1) - (7.3) hold. Then there is a partition of L in three (distributive) sublattices L_0, L_1 and L_2 defined by

$L_0 = \{ x \in L : x \leq a \}$,

$L_1 = \{ x \in L : b \leq x \}$, and

$L_2 = \{ x \in L : a < x < b \}$.

By (7.3), L_2 is a linear ordering. By (6.6), fix an embedding g_0 of L_0 into $\mathcal{L}$ which preserves the least element and an embedding g_1 of L_1 into $\mathcal{L}$ which preserves the greatest element. By the density theorem, choose an embedding f_2 of L_2 into $(\underset{\sim}{c},\underset{\sim}{b})$.
Then the function f from L into $\underset{\sim}{R}$ defined by

$$\forall x \in L: \quad f(x) = \begin{cases} f_0(g_0(x)) & \text{if } x \in L_0 \\ f_1(g_1(x)) & \text{if } x \in L_1 \\ f_2(x) & \text{if } x \in L_2 \end{cases}$$

is an embedding of L into the r.e. degrees which preserves greatest and least elements.

Proof of theorem 7.1. The proof combines that of theorem 6.1 with the Thomason construction [32].

We will construct uniformly r.e. sets A_i, B, and C_i for $i < \omega$, and define, for $\alpha \in \mathcal{L}$,

$$A_\alpha = \bigcup_{i \in \alpha} A_i \qquad \text{and} \qquad C_\alpha = \bigcup_{i \in \alpha} C_i.$$

We set $\underset{\sim}{c} = dg(C_\omega)$ and $\underset{\sim}{b} = dg(C_\omega \oplus B)$ and we will show that the functions f_0 and f_1 defined by

$$f_0(\alpha) = dg(C_\alpha) \quad \text{and} \quad f_1(\alpha) = dg(A_\alpha \oplus B \oplus C_\omega)$$

are embeddings of $\mathcal{L}$ into $[\underset{\sim}{0}, \underset{\sim}{c}]$ and $[\underset{\sim}{b}, \underset{\sim}{0}']$, respectively, which preserve greatest and least elements.

If we make all sets A_i and C_i, $i < \omega$, pairwise disjoint, then f_0 and f_1 will be the desired embeddings if we will meet, for all $\alpha, \beta \in \mathcal{L}$ and $e, i < \omega$ the requirements

$$R_{2\langle e,i\rangle}: \ C_i \neq \{e\},$$

$$R_{2\langle e,i\rangle+1}: \ A_i \neq \{e\}^B \oplus C_\omega,$$

$$N_{2\langle e,\alpha,\beta\rangle}: \ \{e\}^{C_\alpha} = \{e\}^{C_\beta} \text{ total} \implies \{e\}^{C_\alpha} \text{ recursive in } C_{\alpha\cap\beta},$$

$$N_{2\langle e,\alpha,\beta\rangle+1}: \ \{e\}^{A_\alpha \oplus B \oplus C_\omega} = \{e\}^{A_\beta \oplus B \oplus C_\omega} \text{ total} \implies$$

$$\implies \{e\}^{A_\alpha \oplus B \oplus C_\omega} \text{ recursive in } A_{\alpha\cap\beta} \oplus B \oplus C_\omega,$$

and if we guarantee that

$$(7.6) \quad K \leq_T A_\omega \oplus B \oplus C_\omega,$$

where K is some fixed complete set.

(That this suffices is seen as in the proof of theorem 6.1 and in

Thomason [32] .)

It remains to insure that $\underset{\sim}{c} < \underset{\sim}{b}$. By definition of $\underset{\sim}{c}$ and $\underset{\sim}{b}$, $\underset{\sim}{c} \leq \underset{\sim}{b}$. This is sufficient by (6.5).

Let g be a one-one recursive function which enumerates K.

Condition (7.6) is satisfied by the use of markers (as in the proof of theorem 6.1). A recursive function χ is defined during the construction such that

(7.7) $\forall s \; (\; \chi(g(s),s) \neq \chi(g(s),s+1) \;)$,

(7.8) $\forall x \; (\; \lim_s \chi(x,s) < \omega \;$ exists $)$, and

(7.9) $\forall x,s \; (\; \chi(x,s) \neq \chi(x,s+1) \implies$

$\implies \exists y \leq \chi(x,s) \; (\; y \in (A_{\omega,s+1} \setminus A_{\omega,s}) \cup (B_{s+1} \setminus B_s) \cup (C_{\omega,s+1} \setminus C_{\omega,s})))$

It is easily seen that (7.7) through (7.9) imply (7.6).

To the negative requirements N_f, $f < \omega$, a recursive length function is defined by

$$l(2\langle e,\alpha,\beta\rangle,s) = \max \{ x : \forall y < x \; (\; \{e\}_s^{C_\alpha,s}(y) = \{e\}_s^{C_\beta,s}(y) \downarrow \;) \}$$

and

$$l(2\langle e,\alpha,\beta\rangle+1,s) = \max \{ x : \forall y < x \; (\; \{e\}_s^{A_\alpha,s} \oplus B_s \oplus C_{\omega,s}(y) = \{e\}_s^{A_\beta,s} \oplus B_s \oplus C_{\omega,s}(y) \downarrow \;) \} .$$

(As in the proof of theorem 6.1 we may assume that the indices of the negative requirements are numbers.)

Based on this definition of l, the restraint function r and e-maximal stages are defined as in the proof of theorem 1.1.

The number x requires attention at stage s+1 if $x < g(s)$ and there exist numbers $e,i < \omega$ and $j \leq 1$ such that

$$x = \langle 2\langle e,i\rangle+j, r(2\langle e,i\rangle+j,s)\rangle$$

and the following two conditions hold:

(7.10) If $j=0$ then $\{e\}_s(\chi(x,s)) = 0$ and

$\exists y$ ($\{e\}_s(y) = 0$ and $y \in C_{i,s}$).

(7.11) If $j=1$ then $\{e\}_s^{B_s} \oplus C_{\omega,s}(\gamma(x,s)) = 0$ and $\gamma(x,s) \notin A_{i,s}$.

Construction.

Stage s=0. For all $x < \omega$, $\gamma(x,0) = \langle 0,x \rangle$.

Stage s+1. Distinguish the following two cases:

Case 1 : No number requires attention at stage s+1. Then put $\gamma(g(s),s)$ into B and set (for all $y < \omega$)

$$\gamma(y,s+1) = \begin{cases} \gamma(y,s) & \text{if } y < g(s) \\ \langle y,s+1 \rangle & \text{otherwise.} \end{cases}$$

Case 2 : Otherwise. Let x be the least number which requires attention at stage s+1. (We say: x receives attention at stage s+1.) Fix $e,i < \omega$ and $j \leq 1$ such that

$x = \langle 2\langle e,i \rangle + j, r(2\langle e,i \rangle + j,s) \rangle$.

If $j = 0$, put $\gamma(x,s)$ into C_i, otherwise into A_i. In either case set (for all $y < \omega$)

$$\gamma(y,s+1) = \begin{cases} \gamma(y,s) & \text{if } y \leq x \\ \langle y,s+1 \rangle & \text{otherwise.} \end{cases}$$

The construction is effective and thus the sets A_i, B and C_i ($i < \omega$) are uniformly r.e..

Lemma 7.1. For all $x,y,s,t < \omega$:

(a) $\exists s' \leq s$ ($\gamma(x,s) = \langle s',x \rangle$)

(b) $x \neq y \implies \gamma(x,s) \neq \gamma(y,t)$

(c) $x \leq \gamma(x,s) \leq \gamma(x,s+1)$

(d) $x < y \iff \gamma(x,s) < \gamma(y,s)$

(e) $\lim_s \gamma(x,s) < \omega$ exists.

Proof. This is proved as lemma 6.1.

Let $\gamma(x) = \lim_s \gamma(x,s)$ for all $x < \omega$.

Note that lemma 7.1 implies that the sets in $\{A_i : i < \omega\} \cup \cup \{C_i : i < \omega\}$ are all pairwise disjoint.

Lemma 7.2. $K \leq_T A_\omega \oplus B \oplus C_\omega$.

Proof. Condition (7.8) is true by lemma 7.1(e). (7.7) and (7.8) follow from the construction immediately. As we have seen, this suffices to show lemma 7.2.

Lemma 7.3. For all $e < \omega$, $\liminf_s r(e,s) < \omega$.

Lemma 7.3 is proved as lemma 1.1.

For $e = -1$ and all $s < \omega$ set $r(-1,s) = 0$. Further, for $e \geq -1$, set $r(e) = \liminf_s r(e,s)$ and $S_e = \{s : r(e,s) = r(e)\}$.

Lemma 7.4. Every number requires attention at most finitely often.

Proof by induction. Fix $x = \langle 2e+j, y\rangle$, where $j \leq 1$. If $j = 0$ and x receives attention at a stage s' then x doesn't require attention after stage s'. By induction hypothesis, this implies that x requires attention only finitely often. If $j = 1$, the claim is seen as in the proof of lemma 6.4.

Lemma 7.5. For all $f < \omega$, R_f is met.

Proof. Fix $f,e,i < \omega$ and $j \leq 1$ s.t. $f = 2\langle e,i\rangle + j$.

Case 1 : $j = 0$. Assume for a contradiction that R_f is not met, i.e. that $C_i = \{e\}$.

Note that R_f is met if it receives attention at any stage. Hence R_f never receives attention. Since $C_i = \{e\}$, it follows

(7.12) $\forall x,s \; (\; C_i(\gamma(\langle 2\langle e,i\rangle, x\rangle, s)) = \{e\}(\gamma(\langle 2\langle e,i\rangle, x\rangle, s)) = 0 \;)$.

and

(7.13) $\forall x,s\ (\ \{e\}_s(x) = 0 \ \Rightarrow\ x \notin C_{i,s}\ )$.

By (7.12) and (7.13), there are infinitely many stages s such that $r(f,s) = r(f)$, $\chi(\langle f,r(f)\rangle,s) = \chi(\langle f,r(f)\rangle)$, and $\langle f,r(f)\rangle$ requires attention at stage s+1. By lemma 7.4, $\langle f,r(f)\rangle$ eventually receives attention at such a stage s+1. Hence $\chi(\langle f,r(f)\rangle;s) \in C_i$ contrary to (7.12).

Case 2 : $j = 1$. In this case one sees as in the proof of lemma 6.5 that $\chi(\langle 2\langle e,i\rangle+1,r(2\langle e,i\rangle+1)\rangle) \in A_i$ iff $\{e\}^B \oplus C_\omega(\chi(\langle 2\langle e,i\rangle+1,r(2\langle e,i\rangle+1)\rangle)) = 0$.

Lemma 7.6. For all $e,f,s < \omega$:

(a) $e \leq f \ \Rightarrow\ r(e,s) \leq r(f,s)$ and $r(e) \leq r(f)$

(b) $e \leq f \ \Rightarrow\ \exists s_0 \forall s \geq s_0\ (\ s \in S_f \ \Rightarrow\ s \in S_e\ )$.

Lemma 7.6 is proved as lemma 6.6.

By lemmata 7.3 and 7.6, for all $e < \omega$ we can fix a stage s_e for the following such that

(7.14) $s_e \in S_{e-1}$,

(7.15) $\forall s \geq s_e \forall f \leq e-1\ (\ s \in S_{e-1} \ \Rightarrow\ s \in S_f\ )$,

(7.16) $\forall s \geq s_e \forall f \leq e\ (\ r(f,s) \geq r(f)\ )$, and

(7.17) For $x = \chi(\langle e,r(e)\rangle)+1$,

$A_{\omega,s_e}[x] = A_\omega[x]$, $B_{s_e}[x] = B[x]$, and $C_{\omega,s_e}[x] = C_\omega[x]$.

Lemma 7.7. Let s and t be stages such that $s_e \leq s < t$, s is e-maximal and in S_{e-1}, and t is not e-maximal or no element of S_{e-1}. Then at stage t+1 no number $x < s$ is enumerated in $A_\omega \cup C_\omega$.

Lemma 7.7 is proved as lemma 6.7.

Lemma 7.8. For all $e < \omega$, N_e is met.

Lemma 7.8 is proved as lemma 6.8.

The lemmata 7.2, 7.5 and 7.8 imply that the constructed sets have the desired properties.

A classification of the finite distributive lattices which are embeddable into $\underset{\sim}{R}$ preserving greatest and least elements is not known. The following definition, however, seems to be useful for such a classification.

Definition 7.2. A lattice $L = L(\cup, \cap, \leq)$ is n-connected ($n \geqslant 1$) if L has greatest and least elements, say 1 and 0, respectively, and if there exist $a_i, b_i \in L$ ($i < n$) such that

(i) $a_0 \cap b_0 = 0$,

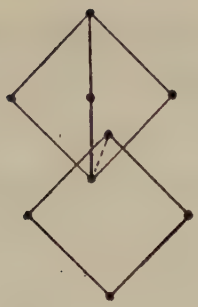
(ii) $\forall i < n-1$ ($a_{i+1} \cap b_{i+1} \leq a_i \cup b_i$);

(iii) $a_{n-1} \cup b_{n-1} = 1$, and

(iv) $\forall i < n$ ($a_i \mid b_i$).

It is easy to see that a finite lattice is connected iff it is n-connected for an $n \geqslant 1$.

By Lachlan's non-diamond theorem no 1-connected lattice is embeddable into $\underset{\sim}{R}$ by a map which preserves greatest and least elements. It is not known if any n-connected lattice is embeddable by such a map. Though we conjecture that this is not the case, we can only show that the following special 2-connected (nondistr.) lattice can't be embedded preserving greatest and least elements:



This is proved by a variant of the non-diamond technique.

By recent results of Jockusch and Soare, one could show that no n-connected lattice ($n \geqslant 1$) can be embedded into $\underset{\sim}{R}$ preserving greatest and least elements if one can prove the following:

(7.18) $\forall \underset{\sim}{a}, \underset{\sim}{b}((\underset{\sim}{a} \in \mathrm{ENC}, \underset{\sim}{b} \notin \mathrm{ENC}$ and $\underset{\sim}{a} \mid \underset{\sim}{b}) \Longrightarrow$

$\Longrightarrow \quad \underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist).

§ 8 ASCENDING SEQUENCES OF UNIFORMLY RECURSIVELY ENUMERABLE DEGREES WITHOUT EXACT PAIRS.

A classical result (Spector [31], Theorem 3) in the theory of Turing degrees is that every ascending sequence of degrees has an exact pair. I.e. if $\{\underset{\sim}{a}_n : n < \omega\}$ is a sequence of degrees s.t. $\forall n\ (\underset{\sim}{a}_n < \underset{\sim}{a}_{n+1})$, then there exist degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ such that $\forall n\ (\underset{\sim}{a}_n < \underset{\sim}{b}_0, \underset{\sim}{b}_1)$ and $\forall \underset{\sim}{c}\ (\underset{\sim}{c} \leq \underset{\sim}{b}_0, \underset{\sim}{b}_1 \Longrightarrow \exists n\ (\underset{\sim}{c} \leq \underset{\sim}{a}_n))$.[1)]
An immediate consequence of this result is that no ascending sequence of degrees has a least upper bound (l.u.b.).
If we choose the degrees $\underset{\sim}{a}_n$ above to be r.e., we may not expect to find an exact pair of r.e. degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ for the sequence $\{\underset{\sim}{a}_n : n < \omega\}$. Using the density theorem, one can easily construct an ascending sequence of r.e. degrees $\underset{\sim}{a}_n$, $n < \omega$, s.t. $\underset{\sim}{0}'$ is the only r.e. degree above all the degrees $\underset{\sim}{a}_n$. Hence in the structure of the r.e. degrees an ascending sequence may have a l.u.b. and thus no exact pair.
For uniformly r.e. sequences, however, the situation is different. No ascending sequence of uniformly r.e. degrees has a l.u.b. (Sacks [16],pp.115/6). Further Yates in [33] constructed such a sequence which has an exact pair.
Shore suggested us to look at the question whether every ascending sequence of uniformly r.e. degrees has an exact pair. We answer the question negatively (Theorem 8.1). Indeed every (nontrivial) initial segment of the r.e. degrees contains such a sequence without exact pair.

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1) In the introduction to §8 (contrary to our convention) $\underset{\sim}{a}, \underset{\sim}{b}$, etc. denote any (not necessarily r.e.) degrees.

Definition 8.1. (a) A sequence $\{\underset{\sim}{a}_n : n<\omega\}$ of r.e. degrees is uniformly recursively enumerable (u.r.e.) if there exists a recursive function f such that $\forall n\ (\ W_{f(n)} \in \underset{\sim}{a}_n\ )$.

(b) R.e. degrees $\underset{\sim}{b}_0$ and $\underset{\sim}{b}_1$ are an exact pair for a sequence $\{\underset{\sim}{a}_n : n<\omega\}$ of r.e. degrees if

(b1) $\forall n\ (\ \underset{\sim}{a}_n < \underset{\sim}{b}_0, \underset{\sim}{b}_1\ )$ and

(b2) $\forall \underset{\sim}{c}\ (\ \underset{\sim}{c} \leq \underset{\sim}{b}_0, \underset{\sim}{b}_1 \Rightarrow \exists n\ (\ \underset{\sim}{c} \leq \underset{\sim}{a}_n\ )\ )$.

Theorem 8.1. There exists an ascending sequence of u.r.e. degrees which has no exact pair.

Proof. The proof builds on Lachlan's nonbranching degree construction in [4] .

We construct an r.e. set A such that

(8.1) $\forall n\ (\ A^{(\leq n)} \leq_T A^{(\leq n+1)}\ )$,

where $A^{(\leq n)} = \bigcup_{i\leq n} A^{(i)}$, and set $\underset{\sim}{a}_n = dg(A^{(\leq n)})$. Then $\{\underset{\sim}{a}_n : n<\omega\}$ is an ascending sequence of uniformly r.e. degrees. To guarantee that the sequence has no exact pair, for every pair of r.e. sets W_{e_0} and W_{e_1}, $e_0, e_1 < \omega$, we construct an r.e. set B_e, $e = \langle e_0, e_1\rangle$, such that

(8.2) $\forall n\ (\ A^{(\leq n)} \leq_T W_{e_0}, W_{e_1}\ ) \Rightarrow$
$\Rightarrow\ (\ B_e \leq_T W_{e_0}, W_{e_1}$ and $\forall n\ (\ B_e \not\leq_T A^{(\leq n)}\ )\ )$.

To satisfy (8.1) it suffices to meet the requirements

$R_{2\langle n,e\rangle}$: $A^{(n+1)} \neq \{e\}^{A^{(\leq n)}}$

for all $n,e < \omega$.

(8.2) will follow if we meet the requirements

$R_{2\langle n,e,i\rangle+1}$: $W_{e_0}, W_{e_1} \not\leq_T A^{(\leq n)} \Rightarrow B_e \neq \{i\}^{A^{(\leq n)}}$

and satisfy the conditions

(8.3) $B_e \leq_T W_{e_o} \oplus A^{(o)}, W_{e_1} \oplus A^{(o)}$

for all n,e,e_o,e_1, and $i < \omega$, where $e = \langle e_o, e_1 \rangle$.

To satisfy (8.3) we use markers γ_j, $j \leq 1$. $\gamma_j(x,s)$ (which denotes the position of the j-marker of x at the end of stage s) is defined by induction on s (for all $x<\omega$) :

(8.4) $\gamma_j(x,0) = \langle 0,j,x,0 \rangle$

(8.5) $$\gamma_j(x,s+1) = \begin{cases} \langle 0,j,x,s+1 \rangle & \text{if } \exists y < x\,(y \in W_{((x)_0)_j,s+1} \setminus W_{((x)_0)_j,s}) \\ \gamma_j(x,s) & \text{otherwise.} \end{cases}$$

Notice that $\lambda x,s(\gamma_j(x,s))$, $j = 0,1$, is a recursive function, $\gamma_j(x,s) \in \omega^{(o)}$, and, for all $x<\omega$, $\lim_s \gamma_j(x,s) < \omega$ exists (by (8.5)). Indeed for $x \in \omega^{(e)}$, $e = \langle e_o, e_1 \rangle$, a stage s_o can be computed recursively in W_{e_j} such that, for all $s \geqslant s_o$, $\gamma_j(x,s) = \gamma_j(x,s_o)$. Hence to guarantee (8.3) it suffices to have $B_e \subseteq \omega^{(e)}$ and to insure that if x enters B_e at stage s, then $\gamma_0(x,s)$ and $\gamma_1(x,s)$ are put into $A^{(o)}$.

The strategy to meet the requirements R_f, $f < \omega$, is that of Friedberg and Muchnik. A restraintfunction r(f,s) for R_f is defined during the construction.

R_f <u>requires attention</u> at stage s+1 if $r(f,s) = 0$ and if there exists a number x such that

(8.6) If $f = 2\langle n,e \rangle$, for some $n,e<\omega$, then $x = \langle n+1,e,x' \rangle$ for some $x'<\omega$, $\{e\}_s^{A_s^{(\leq n)}}(x) = 0$, and $x \geqslant \max\{ r(f',s) : f'< f \}$.

(8.7) If $f = 2\langle n,e,i \rangle+1$, $e = \langle e_o, e_1 \rangle$, for some $n,e,i,e_o,e_1 < \omega$, then $x = \langle e,n,i,x' \rangle$ for some $x'<\omega$, $\{i\}_s^{A_s^{(\leq n)}}(x) = 0$, $x \notin B_{e,s}$, and $\min\{ \gamma_j(x,s) : j \leq 1 \} \geqslant \max\{ r(f',s) : f'< f \} \cup \cup \{ u(i,A_s^{(\leq n)},x,s) \}$.

If R_f requires attention at stage s+1 and x satisfies (8.6) and (8.7) then R_f requires attention through x at stage s+1.

Construction :

Stage 0. For all $f < \omega$, $r(f,0) = 0$.

Stage s+1. If no requirement requires atttention at stage s+1, set $r(f,s+1) = r(f,s)$ for all $f < \omega$.
Otherwise let f be the least index of a requirement which requires attention and let x be the least number through which R_f requires attention. (We say R_f is active at stage s+1.) If f is even put x into A ($A^{(n+1)}$). If f is odd, $f = 2\langle n,e,i\rangle+1$, then put x into B_e and $\gamma_0(x,s+1)$ and $\gamma_1(x,s+1)$ into A ($A^{(o)}$). In either case set

$$r(f',s+1) = \begin{cases} r(f',s) & \text{if } f' < f \\ s+1 & \text{if } f' = f \\ 0 & \text{if } f' > f. \end{cases}$$

Note that the construction is effective and thus A and B_e, $e < \omega$, are r.e. sets. That the sets have the desired properties will follow from the subsequent lemmata.

Lemma 8.1. Every requirement R_f requires attention at most finitely often.

Proof by induction on f. For the inductive step fix an f. By induction hypothesis choose a stage s_0 after which no $R_{f'}$, $f' < f$, requires attention. If R_f requires attention at a stage $s_1 > s_0$, then R_f active at stage s_1 and $r(f,s_1) = s_1$. By choice of s_0

$$\forall s \geq s_1 \ (\ r(f,s) = r(f,s_1) = s_1 > 0 \ )$$

and R_f doesn't require attention after stage s_1.

Lemma 8.2. For all $f < \omega$, $\lim_s r(f,s) < \omega$ exists.

Proof. This follows from lemma 8.1 immediately.

Set $r(f) = \lim_s r(f,s)$.

Lemma 8.3. For all $e,n < \omega$, $R_{2\langle n,e\rangle}$ is met.

Proof. Case 1 : $r(2\langle n,e\rangle) > 0$. Let s be the least stage s.t.

(8.8) $\forall t > s \; (r(2\langle n,e\rangle,t) = r(2\langle n,e\rangle))$.

Then $R_{2\langle n,e\rangle}$ is active at stage s+1. I.e. there exists an $x \in \omega^{(n+1)}$ such that $x \in A^{(n+1)}_{s+1}$, $\{e\}^{A_{s+1}^{(\leq n)}}(x) = \{e\}^{A_s^{(\leq n)}}(x) = 0$, and

(8.9) $r(2\langle n,e\rangle,s+1) = s+1$.

By (8.8) and (8.9), $A_{s+1}[s+1] = A[s+1]$, which implies that the computation $\{e\}^{A_{s+1}^{(\leq n)}}(x)$ is A-correct. Hence $A^{(n+1)}(x) \neq \{e\}^{A^{(\leq n)}}(x)$.

Case 2 : $r(2\langle n,e\rangle) = 0$. Assume for a contradiction that $A^{(n+1)} = \{e\}^{A^{(\leq n)}}$. By lemmata 8.1 and 8.2 and the assumption $r(2\langle n,e\rangle) = 0$, choose s_0 and x_0 such that

$R_{2\langle n,e\rangle}$ doesn't require attention after stage s_0,

$\forall s \geq s_0 \; (r(2\langle n,e\rangle,s) = 0)$,

$x_0 > \max\{ r(f,t) : f < 2\langle n,e\rangle \text{ and } t < \omega \}$, and

$\forall x \geq x_0 \; (\langle n+1,e,x\rangle \notin A)$.

Since $A^{(n+1)} = \{e\}^{A^{(\leq n)}}$,

$\forall x \geq x_0 \; (\{e\}^{A^{(\leq n)}}(\langle n+1,e,x\rangle) = 0)$.

Hence one can choose $x > x_0$ and $s > s_0$ such that $\{e\}^{A_s^{(\leq n)}}(\langle n+1,e,x\rangle) = 0$. Then $R_{2\langle n,e\rangle}$ requires attention through x at stage s+1 contrary to the choice of s_0.

Lemma 8.4. For all $e,n,i < \omega$, $R_{2\langle n,e,i\rangle+1}$ is met.

Proof. Fix $f = 2\langle n,e,i\rangle+1$.

Case 1 : $r(f) > 0$. This case is proved similarly to case 1 in the

the proof of lemma 8.3.

Case 2 : r(f) = 0. Assume for a contradiction that $B_e = \{i\}^{A^{(\leq n)}}$ and $W_{e_0}, W_{e_1} \not\leq_T A^{(\leq n)}$ ($e = \langle e_0, e_1 \rangle$).

By lemmata 8.1 and 8.2 and the assumption r(f) = 0, choose s_0 and x_0 such that

R_f doesn't require attention after stage s_0,

$\forall s \geq s_0 (r(f,s) = 0)$,

$x_0 \geq \max \{ r(f',t) : f' < f \text{ and } t < \omega \}$,

$\forall x \geq x_0 (B_e(\langle e,n,i,x \rangle) = \{i\}^{A^{(\leq n)}}(\langle e,n,i,x \rangle) = 0)$.

(Note that a number $\langle e,n,i,x \rangle$, $x < \omega$, is put into B_e only when R_f is active and that, by assumption, $B_e = \{i\}^{A^{(\leq n)}}$).

Hence there exists an $A^{(\leq n)}$- recursive function $\lambda x(s(x))$ such that for all $x \geq x_0$, $s(x) > s_0$ and

(8.10) $\forall s \geq s(x) (\{i\}_s^{A_s^{(\leq n)}}(\langle e,n,i,x \rangle) = 0$ and $u(i, A_s^{(\leq n)}, \langle e,n,i,x \rangle, s) \leq s(x))$.

Since $W_{e_0} \not\leq_T A^{(\leq n)}$, for infinitely many x

$$W_{e_0,s(x)}[\langle e,n,i,x \rangle] \neq W_{e_0}[\langle e,n,i,x \rangle].$$

Hence there are sets $\{ y_m : m < \omega \}$ and $\{ t_m : m < \omega \}$, which are r.e. in $A^{(\leq n)}$, such that for all $m < \omega$:

$x_0 \leq y_m < y_{m+1}$, $t_m \geq s(y_m)$, and

$$W_{e_0,t_m}[\langle e,n,i,y_m \rangle] \neq W_{e_0,t_m+1}[\langle e,n,i,y_m \rangle].$$

The latter implies that $\gamma_0(\langle e,n,i,y_m \rangle, t_m+1) = \langle 0,0,\langle e,n,i,y_m \rangle, t_m+1 \rangle$, and thus

(8.11) $\forall s > t_m (\gamma_0(\langle e,n,i,y_m \rangle, s) \geq s(y_m))$.

Since $W_{e_1} \not\leq_T A^{(\leq n)}$, for infinitely many m,

$$W_{e_1,t_m}[\langle e,n,i,y_m \rangle] \neq W_{e_1}[\langle e,n,i,y_m \rangle].$$

Fix such an m and a stage $s \geq t_m$ such that

$$W_{e_1,s}[\langle e,n,i,y_m \rangle] \neq W_{e_1,s+1}[\langle e,n,i,y_m \rangle].$$

Then

(8.12) $\gamma_1(\langle e,n,i,y_m\rangle,s+1) = \langle 0,1,\langle e,n,i,y_m\rangle, s+1\rangle \geq s(y_m)$.

By (8.10) - (8.12) and by the choice of s_o, R_f requires attention through $\langle e,n,i,y_m\rangle$ at stage s+1. Since $s+1 > s_o$, this contradicts the properties of s_o.

Lemma 8.5. For all $e > \omega$, $e = \langle e_o,e_1\rangle$, and $j \leq 1$: $B_e \leq_T W_{e_j} \oplus A^{(o)}$.

Proof. Notice that $B_e \subseteq \omega^{(e)}$ and that for all $x \in \omega^{(e)}$

(8.13) $x \in B_e$ iff $\exists s\ (\gamma_o(x,s) \in A^{(o)})$ iff $\exists s\ (\gamma_1(x,s) \in A^{(o)})$.

Hence to decide recursively in $W_{e_j} \oplus A^{(o)}$ $(j \leq 1)$ if $p \in B_e$, for given p, first decide if $p \in \omega^{(e)}$. If not, $p \notin B_e$. Otherwise, by (8.5), recursively in W_{e_j} find a stage s such that

$\forall t > s\ (\gamma_j(p,t) = \gamma_j(p,s))$.

Then by (8.13), $p \in B_e$ iff $A^{(o)} \cap \{\gamma_j(p,s') : s' \leq s\} \neq \emptyset$, and the latter is decidable recursively in $A^{(o)}$.

Lemmata 8.3, 8.4, and 8.5 imply that the set A has the desired properties. □

Note that the above constructed degrees $\underset{\sim}{a}_n$, $n < \omega$, are all non-branching.

The proof of theorem 8.1 can be combined with the permitting method to show that below any nonzero r.e. degree an ascending sequence of u.r.e. degrees without exact pair exists.

Proof. Let $\underset{\sim}{c} > \underset{\sim}{0}$ be given. Fix an r.e. set $C \in \underset{\sim}{c}$ and an effective enumeration $\{C_s : s < \omega\}$ of C. In the definition of "require attention" in (8.6) and (8.7) add the clause

$$\exists y < x\ (y \in C_{s+1} \setminus C_s).$$

Then, by permitting, $A \leq_T C$.

Lemmata 8.1,8.2, and 8.5 hold by the old proofs. The proofs of lemmata 8.3 and 8.4, however, fail in the second case. To argue as before, we need that $C \not\leq_T A$. Hence requirements

$$N_e \quad : \quad C \neq \{e\}^A ,$$

for all $e < \omega$, must be added. These requirements are attacked by the Sacks preservation strategy. A recursive length function $\hat{l}$ and restraint function $\hat{r}$ are defined by

$$\hat{l}(e,s) = \max \{ x : \forall y < x \ (C_s(y) = \{e\}^{A_s}_s (y)) \}, \text{ and}$$

$$\hat{r}(e,s) = \max \{ u(e,A_s,x,s) : x \leq l(e,s) \} .$$

In (8.6) and (8.7) the condition

$$x \geqslant \max \{ \hat{r}(f',s) : f' < f \}$$

must be added. Then after lemma 8.1 or 8.2 the following lemma is inserted :

Lemma 8.2a. For all $e < \omega$:

(a) N_e is met

(b) $\lim_s \hat{r}(e,s) < \omega$ exists.

This lemma is proved by the standard argument (cf Soare [26]). The proofs of the second cases of lemmata 8.3 and 8.4 are modified as follows.

Lemma 8.3 : The proof is again by contradiction. Choose s_0 and x_0 as before. By lemma 8.2a we may assume

(8.14) $\quad x_0 \geqslant \max \{ \hat{r}(f,t) : f < 2^{\langle n,e \rangle} \text{ and } t < \omega \}$.

As before

$$\forall x \geqslant x_0 \ (\{e\}^{A^{(\leq n)}} (\langle n+1,e,x \rangle) = 0).$$

Since $C \not\leq_T A$ by lemma 8.2a, and thus in particular $C \not\leq_T A^{(\leq n)}$, there must be $x > x_0$ and $s > s_0$ such that

$$\{e\}^{A_s^{(\leq n)}}_s (\langle n+1,e,x \rangle) = 0 \quad \text{and} \quad \exists y < \langle n+1,e,x \rangle \ (y \in C_{s+1} \setminus C_s).$$

Then $R_{2^{\langle n,e \rangle}}$ requires attention through $\langle n+1,e,x \rangle$ at stage $s+1$

contrary to the choice of s_o.

Lemma 8.4 : Assume for a contradiction that the requirement is not met and choose s_o and x_o with the properties as before and the additional property (8.14). Define t_m and y_m as before.
Since $W_{e_1} \not\leq_T A^{(\leq n)}$, there exist $A^{(\leq n)}$- r.e. sequences $\{m_k : k<\omega\}$ and $\{s'_k : k<\omega\}$ such that for all $k < \omega$

$$m_k < m_{k+1},\ s'_k \geq t_{m_k},\ \text{and}\ W_{e_1,s'_k}[\langle e,n,i,y_{m_k}\rangle] \neq W_{e_1,s'_{k+1}}[\langle e,n,i,y_{m_k}\rangle].$$

It follows that for all $k < \omega$

$$\forall s > s'_k\ (\ \gamma_1(\langle e,n,i,y_{m_k}\rangle,s) \geq s(y_{m_k})\ ).$$

Since $C \not\leq_T A$, and thus $C \not\leq_T A^{(\leq n)}$, there must be numbers k and s such that

$$\exists y < \langle e,n,i,y_{m_k}\rangle\ (\ y \in C_{s+1} \smallsetminus C_s\ )\ \text{and}\ s \geq s'_k.$$

Then, as easily seen, R_f requires attention through $\langle e,n,i,y_{m_k}\rangle$ at stage s+1 contrary to the choice of s_o. □

§ 9 A GENERALIZATION OF LACHLAN'S NON-DIAMOND THEOREM FOR α-RECURSION THEORY.

Shore [22] proved that for some admissible ordinals α the elementary theory of the α-r.e. degrees is different from that of the r.e. degrees. Indeed, he showed that the Σ_3-theories are different; i.e. there is a sentence which can be expressed by a Σ_3-formula in the first order language of partial orderings, and which is true for the r.e. degrees but false for the α-r.e. degrees for some admissible α's. On the other hand, Lerman [9] proved that all admissible α have the same Σ_1-theory. We close the gap by showing that the Σ_2-theories of some α's are different from that of $\underset{\sim}{R}$ (Corollary 9.5).

We also show that for all admissible ordinals α the α-r.e. degrees don't form a lattice (Corollary 9.3). This generalizes a result of Lachlan [4] and Yates [33] for $\alpha = \omega$.

Both results follow from the main theorem of this section that the extension (4.6) of the non-diamond theorem, which we proved in § 4, is true for the α-r.e. degrees for all admissible ordinals α (Theorem 9.1).

A further consequence is that the partition of the α-r.e. degrees in the cuppable and noncuppable degrees has some nice properties for some α's (Corollary 9.6), which this partition has not in the case $\alpha = \omega$ (Corollary 9.7). Indeed, no partition of $\underset{\sim}{R}$ with that properties is known.

We start with a summary of necessary basic facts of α-recursion theory. If not remarked otherwise, proofs of them can be found in Hinman [3] or Simpson [24].

An ordinal α is called (Σ_1-) <u>admissible</u> if $L(\alpha)$, the α-th level of the constructible hierarchy, satisfies Σ_1-replacement; i.e. if for any function $f : L(\alpha) \longrightarrow L(\alpha)$, if the graph of f is Σ_1 over $L(\alpha)$ and $a \in L(\alpha)$, then $f(a) = \{f(b) : b \in a\} \in L(\alpha)$.

In the following α always denotes an admissible ordinal. Instead of subset of α or (partial) function from α^n ($1 \leq n < \omega$) into α, we often say set or (n-ary) (partial) function. Capital letters A, B, ... denote sets, and lower case letters a, b, ... denote ordinals less than α, with the exception of f, g and h which stand for functions. $\bar{A}$ denotes the set $\alpha \setminus A$. We use the same notation for a set and its characteristic function; i.e. $x \in A$ iff $A(x) = 1$, and $x \notin A$ iff $A(x) = 0$.

A (partial) function is (<u>partial</u>) <u>α-recursive</u> if its graph is Σ_1 over $L(\alpha)$. A set is <u>α-recursively enumerable</u> (<u>α-r.e.</u>) if it is Σ_1 over $L(\alpha)$, <u>α-recursive</u> if its characteristic function is α-recursive, and <u>α-finite</u> if it is an element of $L(\alpha)$. If A is α-finite, and f and B are α-recursive, then $f(A)$ and $A \cap B$ are α-finite.

There exists, for all n s.t. $1 \leq n < \omega$, an n-ary bijective α-recursive function $\lambda x_1, \ldots, x_n (\langle x_1, \ldots, x_n \rangle)$ s.t., for all i ($1 \leq i \leq n$), $x_i \leq \langle x_1, \ldots, x_n \rangle$.

We can fix an α-recursive function k s.t.

(i) $\forall x,y$ ($k(x,y) = 0 \iff x < y$) and

(ii) If A is α-finite, then $A = \{x : k(x,y) = 0\}$ for just one $y < \alpha$.

If $A = \{x : k(x,y) = 0\}$ then we set $A = \mathbb{D}_y$ and call y the canonical index of A. Note that $\{\mathbb{D}_x : x < \alpha\}$ is a list of all α-finite sets without repetition, and that if $y \in \mathbb{D}_x$ then $y < x$. Further, there is an α-recursive function r s.t.

(iii) $\forall x,s,t\ (\ s \leq t \implies \mathbb{D}_{r(s,x)} \subseteq \mathbb{D}_{r(t,x)}\ )$ and

(iv) If A is α-r.e. then $A = \bigcup\{\mathbb{D}_{r(s,x)} : s < \alpha\}$ for some $x < \alpha$.

We set $W_{e,s} = D_{r(s,e)}$ and $W_e = \bigcup\{\mathbb{D}_{r(s,e)} : s < \alpha\}$. Then $\{W_e : e < \alpha\}$ is a standard enumeration of the α-r.e. sets. If $A = W_e$ then e is called an α-r.e. index of A.

W.l.o.g. : $\forall e,x,s\ (\ x \in W_{e,s} \implies x < s\ )$.

$\{A_s : s < \alpha\}$ is an α-effective enumeration of A if $A = \bigcup\{A_s : s < \alpha\}$, $\forall s,t(\ s \leq t \implies A_s \subseteq A_t\ )$, and there is an α-recursive function f such that $A_s = \mathbb{D}_{f(s)}$ for all $s < \alpha$. Obviously, A is α-r.e. iff an α-recursive enumeration of A exists.

A set A is regular if $A \cap x$ is α-finite for all $x < \alpha$.

We set (for all $e,s,x,y < \alpha$ and $A \subseteq \alpha$)

$[e]^A_s(x) = y$ if $\exists u,v(\ \langle x,y,u,v\rangle \in W_{e,s}$, $\mathbb{D}_u \subseteq A$ and $\mathbb{D}_v \subseteq \bar{A}\ )$

and

$\{e\}^A(x) = y$ if $\exists s\ (\ [e]^A_s(x) = y\ )$.

(Note that $\{e\}^A$ might not be a (partial) function.)

A use-function $u(e,A,x)$ of $\{e\}^A$ is defined as follows: If there is no y s.t. $\{e\}^A(x) = y$, set $u(e,A,x) = 0$. Otherwise choose s minimal such that $\exists y\ (\ [e]^A_s(x) = y\ )$ and then choose y,u,v minimal (in this order) s.t. $\langle x,y,u,v\rangle \in W_{e,s}$, $\mathbb{D}_u \subseteq A$ and $\mathbb{D}_v \subseteq \bar{A}$, and set $u(e,A,x) = \max\{u,v,1\}$.

A (partial) function f is weakly-α-recursive in A ($f \leq_{w\alpha} A$) if, for some $e < \alpha$, $f = \{e\}^A$, i.e.

$$\forall x,y \; (\; f(x) = y \iff \{e\}^A(x) = y \;).$$

A set B is weakly-α-recursive in A ($B \leq_{w\alpha} A$) if its characteristic function is weakly-α-recursive in A.

If A is a regular α-r.e. set and $\{A_s : s < \alpha\}$ is an α-effective enumeration of A then (following Maass [12]) we define approximations $\{e\}^{A_s}_s$ and $\lambda x(u(e,A_s,x,s))$ of $\{e\}^A$ and $\lambda x(u(e,A,x))$, respectively, as follows:

If $\neg\exists y \; (\; [e]^{A_s}_s(x) = y \;)$ then $\{e\}^{A_s}_s(x)$ is undefined and $u(e,A_s,x,s) = 0$. Otherwise choose the least t,y,u and v (in this order) such that $\langle x,y,u,v\rangle \in W_{e,t}$, $\mathbb{D}_u \subseteq A_s$, and $\mathbb{D}_v \subseteq \overline{A_s}$ (note that $t \leq s$). Set $\{e\}^{A_s}_s(x) = y$ and $u(e,A_s,x,s) = \max\{u,v,1\}$. Then $\lambda x,s(\{e\}^{A_s}_s(x))$ is a partial α-recursive function whose domain is α-recursive and $\lambda x,s(u(e,A_s,x,s))$ is an α-recursive function. Further, if $\{e\}^A$ is a partial function and $\{e\}^A(x) = y$, then $\lim_s \{e\}^{A_s}_s(x) = y$ (i.e. $\exists t \forall s \; (\; t \leq s < \alpha \rightarrow \{e\}^{A_s}_s(x) = y \;)$) and $\lim_s u(e,A_s,x,s) = u(e,A,x)$.

Note that $u(e,A_s,x,s) \leq s$.

We say that $\{e\}^{A_s}_s(x)$ is B-correct if it is defined and $\{z : z \in u(e,A_s,x,s) \setminus A_s\} \subseteq \overline{B}$.

If $\{e\}^{A_s}_s(x)$ is A-correct (and $\{e\}^A$ is a partial function) then

$$\forall t \geq s \; (\; \{e\}^{A_t}_t(x) = \{e\}^A(x) \text{ and } u(e,A_t,x,t) = u(e,A,x) \;)$$

and (for any set B)

$$A \cap u(e,A_s,x,s) = B \cap u(e,A_s,x,s) \implies \{e\}^A(x) = \{e\}^B(x).$$

We say A is α-recursive in B ($A \leq_\alpha B$) if there exists an $e < \alpha$ such that for all $x < \alpha$:

$$\mathbb{D}_x \subseteq A \text{ iff } \exists u,v \; (\langle x,1,u,v\rangle \in W_e, \; \mathbb{D}_u \subseteq B, \text{ and } \mathbb{D}_v \subseteq \overline{B} \;)$$

and

$\mathbb{D}_x \subseteq \overline{A}$ iff $\exists u,v\ (\langle x,0,u,v\rangle \in W_e,\ \mathbb{D}_u \subseteq B,$ and $\mathbb{D}_v \subseteq \overline{B}\ )$.

A function f is α-recursive in B ($f \leq_\alpha B$) if its (coded) graph is α-recursive in B.

The relation $\leq_\alpha$ is reflexive and transitive. We write $A \equiv_\alpha B$ if $A \leq_\alpha B$ and $B \leq_\alpha A$, and $A \mid_\alpha B$ if $A \not\leq_\alpha B$ and $B \not\leq_\alpha A$.

The set $\{ B : A \equiv_\alpha B \}$ is called the α-degree of A and is denoted by $dg_\alpha(A)$. We say that $dg_\alpha(A)$ is α-recursively enumerable (α-r.e.) if it contains an α-r.e. set.

$\underset{\sim}{a}$, $\underset{\sim}{b}$, ... denote α-r.e. degrees. Define $\underset{\sim}{a} \leq \underset{\sim}{b}$ iff there are sets $A \in \underset{\sim}{a}$ and $B \in \underset{\sim}{b}$ such that $A \leq_\alpha B$. $\underset{\sim}{0}$ denotes the α-degree of the α-recursive sets, and $\underset{\sim}{0}'$ is the α-degree of the α-r.e. set $\{ \langle e,x\rangle : x \in W_e \}$, i.e. the degree of the α-complete sets. Obviously, $\underset{\sim}{0} \leq \underset{\sim}{a} \leq \underset{\sim}{0}'$ for every α-r.e. degree.

Following Shore [21] we define

$W_e^A = \{ x : \exists u,v(\langle x,u,v\rangle \in W_e,\ \mathbb{D}_u \subseteq A,$ and $\mathbb{D}_v \subseteq \overline{A}\ )$,

and say that B is α-r.e. in A if $B = W_e^A$ for some $e < \alpha$. Hence $\{ W_e^A : e < \alpha \}$ is a list of the sets which are α-r.e. in A. $A' = \{ \langle e,x\rangle : x \in W_e^A \}$ is called the jump of A. If B is α-r.e. in A then $B \leq_\alpha A'$, and if $B \leq_\alpha A$ then $B' \leq_\alpha A'$. Hence the jump $\underset{\sim}{a}' = dg_\alpha(A')$ of an α-degree $\underset{\sim}{a} = dg_\alpha(A)$ is welldefined. (Note that, for α-recursive A, $dg_\alpha(A') = \underset{\sim}{0}'$. Hence the definition of $\underset{\sim}{a}'$ is compatible with the earlier one of $\underset{\sim}{0}'$.)

A set A is called low if $dg_\alpha(A') = \underset{\sim}{0}'$. An α-degree $\underset{\sim}{a}$ is low if $\underset{\sim}{a}' = \underset{\sim}{0}'$, i.e. if the sets $\underset{\sim}{a}$ contains are low.

Shore proved (in [22]) that every low α-r.e. set is hyperregular, where a set A is called hyperregular if, for all functions f and all ordinals $x < \alpha$:

$f \leq_{w\alpha} A \implies f(x)$ bounded below α (i.e. $\exists y \forall z < x\ (\ f(z) \leq y\ )$.
Sacks and Simpson [17] showed that every hyperregular α-r.e. set is regular, and that every α-r.e. degree contains a regular α-r.e. set. Further, if A is a hyperregular α-r.e. set and $f \leq_{w\alpha} A$ then $f \leq_{\alpha} A$.

An ordinal $x < \alpha$ is an <u>α-cardinal</u> if there is no α-finite 1-1 function from x in any $y < x$. x is called <u>even</u> if $x = y+2z$ for a limit ordinal y and a $z < \omega$. Otherwise x is called <u>odd</u>.

The following four lemmata provide some further facts we will need.

<u>Lemma 9.1</u>. Let A be an α-r.e. regular set and $\{A_s : s < \alpha\}$ be an α-effective enumeration of A. Then there exists a function f s.t. $f \leq_{w\alpha} A$ and

$$\forall x\ (\ A_{f(x)} \cap x = A \cap x\ ).$$

<u>Proof</u>. Since A is regular, $A \cap x$ is α-finite for all $x < \alpha$. Hence, for every $x < \alpha$, the function g_x defined by

$$g_x(y) = \begin{cases} \mu s(\ y \in A_s\ ) & \text{if } y \in A \cap x \\ 0 & \text{otherwise} \end{cases}$$

is α-recursive. Therefore, by the admissibility of α, $g(A \cap x)$ is bounded below α ; i.e.

$$\forall x \exists s\ (\ A_s \cap x = A \cap x\ )$$

and we can select the least such s for any x by a function f which is weakly-α-recursive in A (If i is an α-r.e. index of the α-r.e. set $\{\langle x,s,u,v\rangle : \forall t < s(\ A_t \cap x \neq A_s \cap x),\ \mathbb{D}_u = A_s \cap x$, and $\mathbb{D}_v = \overline{A_s} \cap x\}$ then $f = \{i\}^A$ will do.).

Lemma 9.2 (Permitting method).

(a) Let A and B be sets, $\{A_s : s < \alpha\}$ an α-effective enumeration of A, $y < \alpha$, and f a function such that $f \leq_{w\alpha} B$ and

(9.1) $\forall x \geq y \; (A_{f(x)} \cap x = A \cap x)$.

Then $A \leq_\alpha B$ and A is regular.

(b) Let A and B be α-r.e. sets with B is regular, $\{A_s : s < \alpha\}$ and $\{B_s : s < \alpha\}$ be α-effective enumerations of A and B, respectively, take $y < \alpha$, and let g be an α-recursive function s.t.

(9.2) $\forall x \geq y \; (B_s[x] = B[x] \Rightarrow A_{g(s,x)}[x] = A[x])$.

Then $A \leq_\alpha B$ and A is regular.

Proof. (a): That A is regular follows from (9.1) immediately. By (9.1), for every $x < \alpha$ and $z = \max\{x,y\}$,

$\mathbb{D}_x \subseteq A$ iff $\mathbb{D}_x \subseteq A_{f(z)}$, and $\mathbb{D}_x \subseteq \overline{A}$ iff $\mathbb{D}_x \subseteq \overline{A_{f(z)}}$.

Since $f \leq_{w\alpha} A$, this implies $A \leq_\alpha B$.

(b): By lemma 9.1, fix a function $f \leq_{w\alpha} B$ s.t.

$\forall x \; (B_{f(x)} \cap x = B \cap x)$.

Define a function h by $h(x) = g(f(x),x)$ for all $x < \alpha$.

Then $h \leq_{w\alpha} B$ and

$\forall x \geq y \; (A_{h(x)} \cap x = A \cap x)$.

Hence A is regular, and by (a), $A \leq_\alpha B$.

Lemma 9.3. If A is a Δ^0_2-set and $\emptyset'$ is a complete α-r.e. set, then $A \leq_{w\alpha} \emptyset'$.

Proof. Let A and $\emptyset'$ be given. Since A is Δ^0_2, there exist α-re-

cursive sets B_i such that, for $i = 0,1$,

$$A(x) = i \iff \exists y \forall z \, (B_i(\langle x,y,z\rangle) = 1).$$

Then

$$A(x) = i \iff \exists y \neg \exists z \, (B_i(\langle x,y,z\rangle) = 0).$$

Define sets C_i, $i = 0,1$, by

$$C_i(\langle x,y\rangle) = i \iff \exists z \, (B_i(\langle x,y,z\rangle) = 0).$$

The sets $\overline{C}_0$ and C_1 are α-r.e., and

$$(9.3) \quad A(x) = i \iff \exists y \, (C_i(\langle x,y\rangle) = 1-i) \quad (i=0,1).$$

Since every α-r.e. set is α-recursive in $\emptyset'$, and hence in particular weakly-α-recursive in $\emptyset'$, there exist α-r.e. sets R_i, $i = 0,1$, such that

$$(9.4) \quad \forall x,y,j \, (C_i(\langle x,y\rangle) = j \iff$$
$$\iff \exists u,v \, (\langle\langle x,y\rangle,j,u,v\rangle \in R_i, \ \mathbb{D}_u \subseteq \emptyset', \text{ and } \mathbb{D}_v \subseteq \overline{\emptyset'})).$$

Then the set R, defined by

$$\forall x,i,u,v \, (\langle x,i,u,v\rangle \in R \iff (i \leq 1 \text{ and } \exists y \, (\langle\langle x,y\rangle,1-i,u,v\rangle \in R_i))),$$

is α-r.e. and, by (9.3) and (9.4), for all x,i

$$A(x) = i \iff \exists u,v \, (\langle x,i,u,v\rangle \in R, \ \mathbb{D}_u \subseteq \emptyset', \text{ and } \mathbb{D}_v \subseteq \overline{\emptyset'}).$$

Hence $A \leq_{wk} \emptyset'$.

Lemma 9.4. Let A be a low α-r.e. set. Then there exists an α-recursive function f such that

$$(9.5) \quad \forall e \, (W_e \cap \{ x : \mathbb{D}_x \subseteq \overline{A} \} \subseteq W_{f(e)}), \text{ and}$$
$$(9.6) \quad \forall e \, (W_e \cap \{ x : \mathbb{D}_x \subseteq \overline{A} \} = \emptyset \implies W_{f(e)} \ \alpha\text{-finite}).$$

Proof. Let $\emptyset'$ be a regular complete α-r.e. set. The set

$$A^+ = \{ x : W_x \cap \{ y : \mathbb{D}_y \subseteq \overline{A} \} \neq \emptyset \}$$

is α-r.e. in A since

$$A^+ = \{ x : \exists y,z \, (\langle x,y,z\rangle \in R, \ \mathbb{D}_y \subseteq A, \text{ and } \mathbb{D}_z \subseteq \overline{A}) \ ,$$

where R is the α-r.e. set $\{\langle x,y,z\rangle : z \in W_x\}$.
Hence $A^+ \leq_\alpha A'$. Since A is low, it follows that $A^+ \leq_\alpha \emptyset'$, and thus $A^+ \leq_{w\alpha} \emptyset'$. Fix $e < \alpha$ s.t. $A^+ = \{e\}^{\emptyset'}$. Since $\emptyset'$ is regular, the α-recursive function g defined by

$$\forall x,s : \quad g(x,s) = \begin{cases} \{e\}_s^{\emptyset'_s}(x) & \text{if } \{e\}_s^{\emptyset'_s}(x) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

has A^+ as its limit, i.e.

$$\forall x \ (\lim_s g(x,s) = A^+(x)).$$

Set $V_{e,s} = \{x : x \in W_{e,s} \text{ and } g(e,s) = 1\}$ and $V_e = \bigcup\{V_{e,s} : s < \alpha\}$. Obviously, $\{\bigcup\{V_{e,t} : t \leq s\} : s < \alpha\}$ is an α-effective enumeration of V_e, which depends uniformly on e. Hence the sets V_e are uniformly α-r.e., i.e. there is an α-recursive function f s.t. $W_{f(e)} = V_e$.
We show that this function f satisfies conditions (9.5) and (9.6). To see (9.5), take $x \in W_e$ s.t. $\mathbb{D}_x \subseteq \bar{A}$. Then $e \in A^+$, so $\lim_s g(e,s) = 1$. Hence there exists $s < \alpha$ s.t. $x \in W_{e,s}$ and $g(e,s) = 1$, i.e. such that $x \in V_{e,s}$. Since $V_e = W_{f(e)}$, then $x \in W_{f(e)}$.
To see (9.6), assume that $W_e \cap \{x : \mathbb{D}_x \subseteq \bar{A}\} = \emptyset$. Then $e \notin A^+$, i.e. there exists an $s_0 < \alpha$ s.t.

$$\forall s \geq s_0 \ (g(e,s) = 0).$$

Hence $W_{f(e)} = V_e = \bigcup\{V_{e,s} : s \leq s_0\}$, and $W_{f(e)}$ is α-finite.

Definition 9.1. For all α-r.e. degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

(a) $\underset{\sim}{a}$ is $\underset{\sim}{b}$-cappable if

$$\exists \underset{\sim}{c} \not\leq \underset{\sim}{b} \forall \underset{\sim}{d} \ (\underset{\sim}{d} \leq \underset{\sim}{a}, \underset{\sim}{c} \implies \underset{\sim}{d} \leq \underset{\sim}{b}).$$

Otherwise $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable.

(b) $\underset{\sim}{a}$ is (non)cappable if $\underset{\sim}{a}$ is (non-)$\underset{\sim}{0}$-cappable.

(d) $\underset{\sim}{a}$ is cuppable if $\exists \underset{\sim}{c} < \underset{\sim}{0}' (\underset{\sim}{0}' = \underset{\sim}{a} \cup \underset{\sim}{c})$. Otherwise

$\underset{\sim}{a}$ is noncuppable.

Let $NC^{\underset{\sim}{a}}$ (NC) denote the set of the non-$\underset{\sim}{a}$-cappable (noncappable) α-degrees.

Theorem 9.1. Let $n < \omega$ and let $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$, $i \leq n$, be α-r.e. degrees such that $\underset{\sim}{b}_0 \cup \ldots \cup \underset{\sim}{b}_n$ is low. Then either $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$ or $\underset{\sim}{a}_i \in NC^{\underset{\sim}{b}_i}$ for some $i \leq n$.

This theorem extends Lerman's result [10] that Lachlan's non-diamond theorem holds for all admissible α.

We will only prove the special case n=1 of the theorem. To get the general result, this proof must be modified similarly to the way the proof of theorem 4.1 was modified to show theorem 4.2. The consequences of theorem 9.1 in which we are interested all follow from the case n=1.

Proof of theorem 9.1. The proof is very similar to that of theorem 4.1, and we will often refer to that proof.

Let $\underset{\sim}{a}_i$, $\underset{\sim}{b}_i$ and $\underset{\sim}{c}_i$ be any α-r.e. degrees such that $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$ is low and $\underset{\sim}{c}_i \not\leq \underset{\sim}{b}_i$ ($i = 0,1$). It suffices to show that one of the following conditions holds:

(9.7) $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 < \underset{\sim}{0}'$

(9.8) $\exists \underset{\sim}{e}$ ($\underset{\sim}{e} \leq \underset{\sim}{a}_0, \underset{\sim}{c}_0$ and $\underset{\sim}{e} \not\leq \underset{\sim}{b}_0$)

(9.9) $\exists \underset{\sim}{e}$ ($\underset{\sim}{e} \leq \underset{\sim}{a}_1, \underset{\sim}{c}_1$ and $\underset{\sim}{e} \not\leq \underset{\sim}{b}_1$).

Choose regular α-r.e. sets $A_i \in \underset{\sim}{a}_i$, $B_i \in \underset{\sim}{b}_i$ and $C_i \in \underset{\sim}{c}_i$ s.t. A_0 and B_0 consist of only even ordinals and A_1 and B_1 consist of only odd ordinals ($i = 0,1$). Fix α-effective enumerations

$\{A_{i,s} : s < \alpha\}$, $\{B_{i,s} : s < \alpha\}$, and $\{C_{i,s} : s < \alpha\}$ of A_i, B_i and C_i, respectively ($i = 0,1$). Set $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. Obviously, A and B are regular α-r.e. sets whose α-degrees are $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$ and $\underset{\sim}{b}_0 \cup \underset{\sim}{b}_1$, respectively. $\{A_s : s < \alpha\}$ and $\{B_s : s < \alpha\}$, where $A_s = A_{0,s} \cup A_{1,s}$ and $B_s = B_{0,s} \cup B_{1,s}$, are α-effective enumerations of A and B, respectively.

Witness-sets for the conditions (9.7) - (9.9) are defined analogously to witness-sets for the conditions (4.10) - (4.12) in the proof of theorem 4.1. A set D is a witness for (9.7), if D is Δ^0_2 and $D \not\leq_{wr} A$. By lemma 9.3 the existence of a witness for (9.7) implies that (9.7) holds. A witness for (9.8) ((9.9)) is an α-r.e. set E s.t. , for $i = 0$ ($i = 1$), $E \leq_\alpha A_i, C_i$ and $E \not\leq_{wr} B_i$. Since $E \not\leq_{wr} B_i$ implies that $E \not\leq_\alpha B_i$, the existence of a witness for (9.8) ((9.9)) shows that (9.8) ((9.9)) holds. (Note that, by the lowness of B_i, B_i is regular and hyperregular, and therefore $E \leq_\alpha B_i$ iff $E \leq_{wr} B_i$. The requirement $E \not\leq_{wr} B_i$ is thus not too strong.)

Hence to prove the theorem it suffices to get a witness-set for one of the conditions (9.7) - (9.9). This is done as in the proof of theorem 4.1. We will construct a set D (for (9.7)) and sequences $\{E_e : e < \alpha\}$ (for (9.8)) and $\{E_{(e,i)} : e,i < \alpha\}$ (for (9.9)) of sets, we will satisfy the conditions

(9.10) D is Δ^0_2,

(9.11) $\forall e,i$ (E_e and $E_{(e,i)}$ are α-r.e.),

(9.12) $\forall e,i$ ($E_e \leq_\alpha A_0$ and $E_{(e,i)} \leq_\alpha A_1$),

(9.13) $\forall e$ ($D = \{e\}^A \implies E_e \leq_\alpha C_0$), and

(9.14) $\forall e,i$ (($D = \{e\}^A$ and $E_e = \{i\}^{B_0}$) $\implies E_{(e,i)} \leq_\alpha C_1$),

and we will meet the requirements

R_e : $D \neq \{e_0\}^A$ or $E_{e_0} \neq \{e_1\}^{B_0}$ or $E_{(e_0,e_1)} \neq \{e_2\}^{B_1}$

for all $e,e_0,e_1,e_2 < \alpha$ s.t. $e = \langle e_0,e_1,e_2\rangle$.
As in the proof of theorem 4.1 it is seen that this suffices to make one of the constructed sets a witness for its corresponding condition (and thus to prove the theorem).

By lemma 9.4 we can choose an α-recursive function f such that

(9.15) $\forall e\ (\ W_e \cap \{x : \mathbb{D}_x \subseteq \overline{B}\} \subseteq W_{f(e)}\ )$ and

(9.16) $\forall e\ (\ W_e \cap \{x : \mathbb{D}_x \subseteq \overline{B}\} = \emptyset \Rightarrow W_{f(e)}$ α-finite).

Let $\lambda x,y([x]_y)$ be an α-recursive function s.t., for all $x,y < \alpha$, $W_{[x]_y} = \{z : \langle y,z\rangle \in W_x\}$, and let D_s, $s < \alpha$, be the set of ordinals which are put into D at a stage $\leq s$ and not extracted from D at a later stage $\leq s$ in the construction.

As in §4 the construction depends on n ($n < \alpha$). The <u>n-construction</u> is the same as that in the proof of theorem 4.1 with the following exceptions: In (4.17), (4.18) and step 4 replace $C_{1,s} \setminus C_{1,s-1}$, $C_{0,s} \setminus C_{0,s-1}$ and $A_s \setminus A_{s-1}$ by $C_{1,s} \setminus \bigcup_{t<s} C_{1,t}$, $C_{0,s} \setminus \bigcup_{t<s} C_{0,t}$ and $A_s \setminus \bigcup_{t<s} A_t$, respectively. Everywhere replace ω by α and number by ordinal less than α.

The n-construction is α-effective uniformly in n, $n < \alpha$. Hence the sets V enumerated in these constructions are uniformly α-r.e.. By the recursion theorem we can fix an $n < \alpha$ s.t. the set V of the n-construction has the α-r.e. index n, i.e. $V = W_n$. Fix such an n for the remainder of the proof. We show that this n-construction works.

First note that $\{E_{e,s} : s < \alpha\}$ and $\{E_{(e_0,e_1),s} : s < \alpha\}$ are α-effective enumerations of E_e and $E_{(e_0,e_1)}$, respectively (for all $e,e_0,e_1 < \alpha$).

The following lemmata show that the constructed sets have the de-

sired properties.

Lemma 9.5. Every stage of the n-construction is completed after finitely many steps.

Proof. As in the proof of lemma 4.1, using (9.15) in place of (4.15).

Lemma 9.6. E_{e_0} and $E_{(e_0,e_1)}$ are α-r.e., $E_{e_0} \leq_\alpha A_0$ and $E_{(e_0,e_1)} \leq_\alpha A_1$ (for all $e_0,e_1 < \alpha$).

Proof. That E_{e_0} and $E_{(e_0,e_1)}$ are α-r.e. follows from the α-effectiveness of the n-construction.

As in the proof of lemma 4.1, one sees that for all x,s

$$x \in E_{e_0,s+1} \setminus E_{e_0,s} \implies \exists y \leq x \, (\, y \in A_{0,s} \setminus \bigcup_{t<s} A_{0,t} \,),$$

i.e.

$$A_{0,s} \cap x = A_0 \cap x \implies E_{e_0,s+1} \cap x = E_{e_0} \cap x.$$

Since A_0 is regular, this implies that $E_{e_0} \leq_\alpha A_0$ by lemma 9.2. $E_{(e_0,e_1)} \leq_\alpha A_1$ is shown similarly.

We call an A-information u correct (correct at stage s+1) if $A \cap \mathbb{D}_u = \emptyset$ ($A_s \cap \mathbb{D}_u = \emptyset$). The B-information v is correct (correct at stage s+1) if $B \cap \mathbb{D}_v = \emptyset$ ($B_s \cap \mathbb{D}_v = \emptyset$). The B-information which an attack has at the end of the stage where it is advanced is called final.

Lemma 9.7. For all $e < \alpha$ there exists a stage $s < \alpha$ after which no attack on R_e is advanced.

Proof. As in the proof of lemma 4.3, we can assume that every attack on R_e is eventually cancelled, and that $W_{f([n]_e)}$ is α-finite. Further, if an attack on R_e is advanced at a stage s+1 and v is the final B-information of the attack, then $v \in W_{f([n]_e)}$, and v is correct at stage s+1 but not correct.

Since $W_{f([n]_e)}$ is α-finite there exists some $x < \alpha$ which bounds $W_{f([n]_e)}$, and since B is regular there is a stage $s_o < \alpha$ s.t. $B_{s_o} \cap x = B \cap x$. Hence, by the above facts on the final B-information of an attack on R_e, no attack on R_e is advanced at a stage after s_o.

Lemma 9.8. D is a Δ^o_2-set.

Proof. Since the relation $x \in D_s$ is α-recursive, this follows from lemma 9.7 as lemma 4.4 followed from lemma 4.3.

Lemma 9.9. If $D = \{e_o\}^A$ and $e = \langle e_o, e_1, e_2 \rangle$, then every advanced attack on R_e is eventually completed or cancelled (for all $e, e_o, e_1, e_2 < \alpha$).

The proof of lemma 9.9 is the same as that of lemma 4.5.

Lemma 9.10. Let $D = \{e_o\}^A$, $E_{e_o} = \{e_1\}^{B_o}$ and $e = \langle e_o, e_1, e_2 \rangle$ (for any $e, e_o, e_1, e_2 < \alpha$). Then every attack on R_e is eventually completed or cancelled.

Proof. If we can show that there exists a function g which is weakly-α-recursive in B_o and which satisfies

(++) $\forall x \forall y \leq \langle e, x \rangle$ ($\{e_1\}_{g(x)}^{B_o, g(x)}(y)$ is defined and the computation is B_o-correct),

then, using lemma 9.2(a), we can argue as in the proof of lemma 4.6 to prove the lemma.

By the assumption $E_{e_o} = \{e_1\}^{B_o}$, $\{e_1\}^{B_o}(x)$ is defined for all $x < \alpha$. Hence, by the regularity of B_o,

$\forall x \exists s$ ($\{e_1\}_s^{B_o, s}(x) \downarrow$ and $B_{o,s} \cap u(e_1, B_{o,s}, x, s) = B_o \cap u(e_1, B_{o,s}, x, s)$).

For every $x < \alpha$ we can select the least such s by a function $g' \leq_{w\alpha} B_o$; i.e.

$\forall x$ ($\{e_1\}_{g'(x)}^{B_0,g'(x)}(x) \downarrow$ and

$B_{0,g'(x)} \cap u(e_1, B_{0,g'(x)}, x, g'(x)) = B_0 \cap u(e_1, B_{0,g(x)}, x, g'(x)))$.

(Let i be an α-r.e. index of the α-r.e. set

$\{\langle x,s,y,z\rangle : \{e_1\}_s^{B_0,s}(x)$, $\mathbb{D}_y = B_{0,s} \cap u(e_1, B_{0,s}, x, s)$,

$\mathbb{D}_z = \overline{B_{0,s}} \cap u(e_1, B_{0,s}, x, s)$, and

$\forall t < s(\{e_1\}_t^{B_0,t}(x) \downarrow \implies$

$\implies B_{0,t} \cap u(e_1, B_{0,t}, x, t) \neq B_{0,s} \cap u(e_1, B_{0,t}, x, t))\}$.

Then $g' = \{i\}^{B_0}$ will do.)

By the hyperregularity of B_0,

$\forall x$ ($\sup\{g'(y) : y \leq \langle e,x\rangle\} < \alpha$).

Hence

$\forall x \exists s \forall y \leq \langle e,x\rangle$ ($\{e_1\}_s^{B_0,s}(y) \downarrow$ and

$B_{0,s} \cap u(e_1, B_{0,s}, y, s) = B_0 \cap u(e_1, B_{0,s}, y, s)$).

Again, we can select the least such s for every x by a function $g \leq_{wtt} B_0$. Obviously, g satisfies (++).

Lemma 9.11. If $D = \{e_0\}^A$, then $E_{e_0} \leq_\alpha C_0$ (for all $e_0 < \alpha$).

Proof. Let $D = \{e_0\}^A$. By lemma 9.9 and by the admissibility of α, there exists an α-recursive function g such that every attack on a requirement $R_{\langle e_0,e_1,e_2\rangle}$, $e_1, e_2 < \alpha$, which has a B_0-attacker less than or equal to x and which is advanced at a stage $\leq s+1$, is completed or cancelled before stage g(x,s).

As in the proof of lemma 4.7 we see that

$C_{0,s} \cap x = C_0 \cap x \implies E_{e_0,g(x,s)} \cap x = E_{e_0} \cap x$

for all $x,s < \alpha$. This implies $E_{e_0} \leq_\alpha C_0$ by lemma 9.2.

Lemma 9.12. If $D = \{e_0\}^A$ and $E_{e_0} = \{e_1\}^{B_0}$, then $E_{(e_0,e_1)} \leq_\alpha C_1$ (for all $e_0, e_1 < \alpha$).

Lemma 9.12 follows from lemma 9.10 as lemma 9.11 follows from

lemma 9.9.

Lemma 9.13. For all $e < \alpha$, the requirement R_e is met.

Proof. We can argue as in the proof of lemma 4.9, if we can show that - under the assumption that $E_{(e_0,e_1)} = \{e_2\}^{B_1}$ - there exist a function $g \leq_{wa} B_1$ s.t.

$\forall x \forall y \leq \langle e,x \rangle$ ($\{e_2\}^{B_1}_{g(x)},g(x)(y)$ is defined and the computation is B_1-correct).

The proof of the existence of such a function is similar to the proof of the existence of g in lemma 9.10.

Lemmata 9.6, 9.8, 9.11, 9.12 and 9.13 imply that one of the constructed sets is a witness for its associated condition. As we have seen earlier, this proves the theorem.

Corollary 9.1. Let $n < \omega$ and $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_n$ be α-r.e. degrees s.t. $\underset{\sim}{a}_0$ is low and $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are cappable. Then $\underset{\sim}{a}_0 \cup \ldots \cup \underset{\sim}{a}_n < \underset{\sim}{0}'$.

Proof. As in the proof of corollary 4.1.

The next corollary generalizes a result of Yates [33] in classical recursion theory to α-recursion theory.

Corollary 9.2. There exists an incomplete noncappable α-degree.

Proof. Shore [20] proved that the splitting theorem is valid for the α-r.e. degrees for all admissible ordinals α . Hence there exist incomplete α-r.e. degrees $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ s.t. $\underset{\sim}{a}_1 \cup \underset{\sim}{a}_2 = \underset{\sim}{0}'$. Corollary 9.1 implies thatone of these degrees is noncappable (set $n = 2$ and $\underset{\sim}{a}_0 = \underset{\sim}{0}$).

Corollary 9.3.

(a) If $\underset{\sim}{a} \neq \underset{\sim}{0}$ is a low α-r.e. degree, then there exists an α-r.e. degree $\underset{\sim}{b}$ such that $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist.

(b) The α-r.e. degrees don't form a lattice.

Proof. (a) is proved as Corollary 4.3. (b) follows from (a) since Simpson proved in [23] that, for all admissible α, low nonzero α-r.e. degrees exist.

Corollary 9.4. For some admissible ordinals α all of the following hold:

(a) There exists no embedding of the four-element Boolean algebra into the α-r.e. degrees which preserves the greatest element.

(b) Every cuppable α-degree is noncappable.

(c) If $\underset{\sim}{a}$ is cuppable and $\underset{\sim}{b}$ is noncuppable, then $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable.

(d) The cuppable α-degrees form a filter (in the α-r.e. degrees). I.e.

(d1) If $\underset{\sim}{a}$ is cuppable and $\underset{\sim}{a} \prec \underset{\sim}{b}$, then $\underset{\sim}{b}$ is cuppable.

(d2) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are cuppable and $\underset{\sim}{a} \cap \underset{\sim}{b}$ exists, then $\underset{\sim}{a} \cap \underset{\sim}{b}$ is cuppable.

(e) If $\underset{\sim}{a}$ is cuppable, $\underset{\sim}{b}$ is noncuppable and $\underset{\sim}{a} \mid \underset{\sim}{b}$, then $\underset{\sim}{a} \cap \underset{\sim}{b}$ doesn't exist.

Proof. A sufficient condition for corollary 9.4 to hold for an ordinal α is that all incomplete α-r.e. degrees are low. Shore [21] has shown that α has this property if $\alpha = \lambda^{L}_{\lambda}$, where λ is

a limit ordinal s.t. $\lambda < \aleph_\lambda^L$.

For the following proof of the individual parts of the corollary, fix such an α .

(a). Assume for a contradiction that there exists an embedding of the four-element Boolean algebra into the α-r.e. degrees which preserves greatest element. Then there must exist α-r.e. degrees $\underset{\sim}{a}_0$, $\underset{\sim}{a}_1$ and $\underset{\sim}{b}$ such that $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$, $\underset{\sim}{a}_0 \mid \underset{\sim}{a}_1$ and $\underset{\sim}{b} = \underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$. Both $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ are $\underset{\sim}{b}$-cappable. By theorem 9.1, $\underset{\sim}{b}$ cannot be low, and thus by the choice of α , $\underset{\sim}{b} = \underset{\sim}{0}'$. This contradicts that $\underset{\sim}{b}$ is the infimum of two incomparable α-r.e. degrees.

(b). Let $\underset{\sim}{a}_0$ be cuppable. Then there exists an incomplete, and thus by assumption low, α-r.e. degree $\underset{\sim}{a}_1$ such that $\underset{\sim}{a}_0 \cup \underset{\sim}{a}_1 = \underset{\sim}{0}'$. By the theorem (set $\underset{\sim}{b}_0 = \underset{\sim}{0}$, $\underset{\sim}{b}_1 = \underset{\sim}{a}_1$) either $\underset{\sim}{a}_0$ is noncappable or $\underset{\sim}{a}_1$ is non-$\underset{\sim}{a}_1$-cappable. Since every α-degree $\underset{\sim}{a} \neq \underset{\sim}{0}'$ is $\underset{\sim}{a}$-cappable, $\underset{\sim}{a}_0$ must be noncappable.

(c). Let $\underset{\sim}{a}$ be cuppable and $\underset{\sim}{b}$ be noncuppable. Then there exist a $\underset{\sim}{c} < \underset{\sim}{0}'$ s.t. $\underset{\sim}{a} \cup \underset{\sim}{c} = \underset{\sim}{0}'$. Since $\underset{\sim}{b}$ is noncuppable, $\underset{\sim}{d} = \underset{\sim}{c} \cup \underset{\sim}{b}$ is incomplete and hence low. Obviously, $\underset{\sim}{a} \cup \underset{\sim}{d} = \underset{\sim}{0}'$ and by the theorem (set $\underset{\sim}{a}_0 = \underset{\sim}{a}$, $\underset{\sim}{b}_0 = \underset{\sim}{b}$, $\underset{\sim}{a}_1 = \underset{\sim}{d}$, $\underset{\sim}{b}_1 = \underset{\sim}{d}$) either $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable or $\underset{\sim}{d}$ is non-$\underset{\sim}{d}$-cappable. Since every α-degree $\underset{\sim}{e} \neq \underset{\sim}{0}'$ is $\underset{\sim}{e}$-cappable, it follows that $\underset{\sim}{a}$ is non-$\underset{\sim}{b}$-cappable.

(d). (d1) follows from the definition of cuppable. (d2) follows from (c).

(e). This follows from (c) since the noncuppable α-degrees are closed downwards.

Remark. None of the parts (a) - (e) of corollary 9.4 is true for the r.e. degrees: Shoenfield and Soare [19], and independently Lachlan [7], showed that the four-element Boolean algebra is em-

beddable into $\underset{\sim}{R}$ by a map which preserves the greatest element. Harrington's cup-and-cap theorem (cf. (4.4)) implies that (b) and (c) fail for the r.e. degrees. In theorem 9.2 we will show that the cuppable degrees don't form a filter. Finally (as a consequence of Harrington's anti-cupping and cup-and-cap theorems) there is a minimal pair of which one half is cuppable while the other is noncuppable, which refutes (e) for $\alpha = \omega$.

Corollary 9.5. For some admissible α , the Σ_2-theory of the partial ordering of the α-r.e. degrees is different from that of the r.e. degrees.

Proof. By the previous remark, it suffices to show that at least one of the statements (a) - (e) in corollary 9.4, or its negation, can be expressed by a Σ_2-formula. This is true for (a) and (b). E.g. the following Σ_2-formula asserts the negation of (b):

$$\exists a,b,c,d,e \forall f \, (\, f \leq d \, \wedge \, e \leq f \, \wedge \, d \neq a \, \wedge \, c \neq e \, \wedge$$
$$\wedge \, (\, a,b \leq f \rightarrow d \leq f \,) \, \wedge \, (\, f \leq b,c \rightarrow f \leq e \,) \,).$$

Corollary 9.6. For some admissible α : There exists a non-trivial partition of the α-r.e. degrees in a filter F and an ideal I such that

(9.17) $\forall \underset{\sim}{a}_0, \underset{\sim}{a}_1 \, (\, (\, \underset{\sim}{a}_0 \in F, \, \underset{\sim}{a}_1 \in I, \text{ and } \underset{\sim}{a}_0 \mid \underset{\sim}{a}_1 \,) \Longrightarrow$
$\Longrightarrow \quad \underset{\sim}{a}_0 \cap \underset{\sim}{a}_1$ doesn't exist).

This partition is given by the cuppable and noncuppable degrees.

For the r.e. degrees corollary 9.6 fails. As we will show below the cuppable degrees don't form a filter in $\underset{\sim}{R}$. Also (9.17) fails, as remarked already.

Only recently Jockusch and Soare found a nontrivial partition of $\underset{\sim}{R}$ in a filter and an ideal. If this partition satisfies (9.17) is not known.

The proof of corollary 9.6 requires the following lemma.

Lemma 9.13. The noncuppable α-r.e. degrees form an ideal; i.e.

(i) If $\underset{\sim}{a}$ is noncuppable and $\underset{\sim}{b} \prec \underset{\sim}{a}$, then $\underset{\sim}{b}$ is noncuppable.

(ii) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are noncuppable, then $\underset{\sim}{a} \cup \underset{\sim}{b}$ is noncuppable.

Proof. (i) Trivial.

(ii) Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be noncuppable and assume for a contradiction that $\underset{\sim}{a} \cup \underset{\sim}{b}$ is cuppable. Then there exists a $\underset{\sim}{c} \prec \underset{\sim}{0}'$ s.t. $(\underset{\sim}{a} \cup \underset{\sim}{b}) \cup \underset{\sim}{c} = \underset{\sim}{0}'$. Since $\underset{\sim}{b}$ is noncuppable, $\underset{\sim}{b} \cup \underset{\sim}{c} \prec \underset{\sim}{0}'$. It follows (since $\underset{\sim}{a} \cup (\underset{\sim}{b} \cup \underset{\sim}{c}) = (\underset{\sim}{a} \cup \underset{\sim}{b}) \cup \underset{\sim}{c}$) that $\underset{\sim}{a}$ is cuppable contrary to hypothesis.

Proof of corollary 9.6. Let $\alpha = \aleph_\lambda^L$, where λ is a limit ordinal and $\lambda < \aleph_\lambda^L$, and let F be the set of the cuppable degrees and I be that of the noncuppable degrees. By lemma 9.13, I is an ideal. Since (as we have seen in the proof of corollary 9.4) corollary 9.4 holds for the chosen α , F is a filter and (9.17) is true (by (d) and (e)). Hence it suffices to show that $|I| > 1$ and $|F| > 1$.

By the splitting theorem, there are incomparable α-r.e. degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ such that $\underset{\sim}{0}' = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$. This implies $|F| > 1$.

To show that $|I| > 1$ we have to show that there is a nonzero noncuppable α-degree. By corollary 9.4(b) it suffices to show that there is a nonzero cappable α-degree, i.e. that there exists

a minimal pair of α-r.e. degrees. Lerman and Sacks [11] proved that minimal pairs exist, if there exists no greatest α-cardinal. Since the latter is true for the chosen α , $|I| > 1$.
This completes the proof of corollary 9.6.

We conclude this section with the proof that in classical recursion theory the cuppable degrees don't form a filter.

Theorem 9.2. There exist cuppable degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ which form a minimal pair.

Corollary 9.7. The set of the cuppable degrees is no filter in $\underset{\sim}{R}$.

Proof. By theorem 9.2 there exist cuppable degrees $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ whose infimum is $\underset{\sim}{0}$ and thus noncuppable. Hence the set of the cuppable degrees is not closed under finite infima.

Proof of theorem 9.2. The proof resembles that in §§ 6 and 7.

We construct r.e. sets A_i and B_i ($i = 0,1$) such that

(9.18) $dg(A_0) \cap dg(A_1) = \underset{\sim}{0}$,

(9.19) $\forall i \leq 1 \; (A_i \not\leq_T B_i)$, and

(9.20) $\forall i \leq 1 \; (A_i \oplus B_i$ is complete).

Then $\underset{\sim}{a}_0 = dg(A_0)$ and $\underset{\sim}{a}_1 = dg(A_1)$ form a minimal pair and both are cuppable.

(9.18) and (9.19) will follow if we meet the requirements

N_e : $\{e\}^{A_0} = \{e\}^{A_1}$ total $\Longrightarrow$ $\{e\}^{A_0}$ recursive

and

R_{2e+i} : $A_i \neq \{e\}^{B_i}$

for all $e < \omega$ and $i \leq 1$ (cf. the proof of theorem 1.1).

(9.20) is guaranteed by markers. Let K be a complete set and g be a recursive function which enumerates K without repetitions. Then recursive functions γ_i ($i = 0,1$) are defined during the construction such that

(9.21) $\forall s$ ($\gamma_1(g(s),s) \neq \gamma_i(g(s),s+1)$),

(9.22) $\forall x$ ($\lim_s \gamma_i(x,s) < \omega$ exists), and

(9.23) $\forall x,s$ ($\gamma_i(x,s+1) \neq \gamma_i(x,s) \implies$

$\implies \exists y \leq \gamma_i(x,s)$ ($y \in (A_{i,s+1} \setminus A_{i,s}) \cup (B_{i,s+1} \setminus B_{i,s})$)).

Conditions (9.21) - (9.23) imply that $K \leq_T A_i \oplus B_i$ ($i = 0,1$).

To the requirements N_e a length function l is defined by

$$l(e,s) = \max\{ x : \forall y < x \; (\{e\}_s^{A_{0,s}}(y) = \{e\}_s^{A_{1,s}}(y) \downarrow) \} .$$

Based on this definition a restraint function r and e-maximal stages are defined as in the proof of theorem 1.1.

A number x requires attention at stage s+1, if $x < g(s)$ and if there exist $e < \omega$ and $i \leq 1$, such that $x = \langle 2e+i, r(e,s) \rangle$, $\{e\}_s^{B_{i,s}}(\gamma_i(x,s)) = 0$ and $\gamma_i(x,s) \notin A_{i,s}$.

Construction:

Stage s=0. For all $x < \omega$ and $i \leq 1$ define $\gamma_i(x,0) = \langle 0,x \rangle$.

Stage s+1. Distinguish the following two cases.

Case 1 : No number requires attention at stage s+1. Then, for $i = 0,1$, put $\gamma_i(g(s),s)$ into B_i and set

$$\gamma_i(y,s+1) = \begin{cases} \gamma_i(y,s) & \text{if } y < g(s) \\ \langle s+1, y \rangle & \text{otherwise.} \end{cases}$$

Case 2 : Otherwise. Let x be the least number which requires attention at stage s+1. Choose $e < \omega$ and $i \leq 1$ s.t. $x = \langle 2e+i, r(e,s) \rangle$. Put $\gamma_i(x,s)$ into A_i and $\gamma_{1-i}(g(s),s)$

into B_{1-i}. Define

$$\gamma_i(y,s+1) = \begin{cases} \gamma_i(y,s) & \text{if } y \leq x \\ \langle s+1, y\rangle & \text{otherwise} \end{cases}$$

and

$$\gamma_{1-i}(y,s+1) = \begin{cases} \gamma_{1-i}(y,s) & \text{if } y < g(s) \\ \langle s+1, y\rangle & \text{otherwise.} \end{cases}$$

To see the correctness of the construction, one has to prove the following lemmata. We omit the proofs, which are very similar to the proofs of the analogous lemmata in §§ 6 and 7.

Lemma 9.14. For all $x,y,s,t < \omega$ and $i \leq 1$:

(a) $\exists\, s' \leq s\ (\ \gamma_i(x,s) = \langle s', x\rangle\ )$

(b) $x \neq y \Rightarrow \gamma_i(x,s) \neq \gamma_i(y,t)$

(c) $x \leq \gamma_i(x,s) \leq \gamma_i(x,s+1)$

(d) $x < y \iff \gamma_i(x,s) < \gamma_i(y,s)$

(e) $\lim_s \gamma_i(x,s) \leq \omega$ exists.

Set $\gamma_i(x) = \lim_s \gamma_i(x,s)$ (for all $x < \omega$ and $i \leq 1$).

Lemma 9.15. For $i = 0,1$, $K \leq_T A_i \oplus B_i$.

Lemma 9.16. For all $e < \omega$, $r(e) = \liminf_s r(e,s) < \omega$.

Lemma 9.17. Every number requires attention at most finitely often.

Lemma 9.18. For all $e < \omega$, the requirement R_e is met.

Lemma 9.20. For all $e,f,s < \omega$:

(a) $e \leq f \Rightarrow r(e,s) \leq r(f,s)$ and $r(e) \leq r(f)$.

(b) $e < f \Rightarrow \exists s_0 \forall s \geq s_0\ (\ s \in S_f \Rightarrow s \in S_e\ )$,

where $S_e = \{\, s : r(e,s) = r(e) \,\}$ for all $e < \omega$.

By lemmata 9.16 and 9.20 fix, for all $e < \omega$, a stage s_e s.t.

$s_e \in S_{e-1}$,

$\forall s \geqslant s_e \forall f \leq e-1 \; (s \in S_{e-1} \Rightarrow s \in S_f)$,

$\forall s \geqslant s_e \forall f \leq e \; (r(f,s) \geqslant r(f))$, and

for $i = 0,1$ and $x_i = \chi_i(\langle e, r(e) \rangle)+1$,

$$A_{i,s_e}[x_i] = A_i[x_i] \text{ and } B_{i,s_e}[x_i] = B_i[x_i] .$$

Lemma 9.21. Let s and t be stages such that $s_e < s < t$, s is e-maximal and in S_{e-1}, and t is not e-maximal or no element of S_{e-1}. Then at stage t+1 no number $x < s$ enters $A_0 \cup A_1$.

Lemma 9.22. For all $e < \omega$, N_e is met.

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